HKPhO 2009 Solutions

MC1 (d) $70\vec{x}_0 + 10\vec{y}_0$ MC2 (b) 1.2 MC3 (b) 9.5

MC4 (e) 1.0×10^{-2} mm MC5 (a) 4 times MC6 (c) $\dfrac{mgL\sin\theta}{4}$

MC7 (a) $\left(\dfrac{\sqrt{2}+\sqrt{10}}{4}\right)mgl$ MC8 (c) 40 mC MC9 (b) inward, $\dfrac{Mg}{Qv\cos\theta}$

MC10 (c) $T > T_1 > T_2$ MC11 (a) $\dfrac{1}{2\pi}\sqrt{\dfrac{6g}{H}}$ MC12 (a) 4V

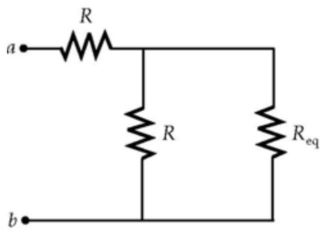
MC13 (e) 1.5 MC14 (a) $\dfrac{2h}{3}$ MC15 (b) $\dfrac{mg}{\sqrt{2}}$ MC16 (e) 1,4,3,2

MC17 (c) **r**, -**v**, **a** MC18 (d) $\tau/2$ MC19 (e) not move

MC20 (c) $\sqrt{u^2 + \dfrac{2}{1.1}GM\left(\dfrac{1}{b^{1.1}} - \dfrac{1}{a^{1.1}}\right)}$

Q1

Let R be the resistance of each resistor in the ladder and let R_{eq} be the equivalent resistance of the infinite ladder. If the resistance is finite and non-zero, then adding one or more stages to the ladder will not change the resistance of the network. We can apply the rules for resistance combination to the diagram shown to the right to obtain a quadratic equation in R_{eq} that we can solve for the equivalent resistance between points a and b.



The equivalent resistance of the series combination of R and ($R \parallel R_{eq}$) is R_{eq}, so:

$$R_{eq} = R + R \parallel R_{eq} = R + \frac{RR_{eq}}{R + R_{eq}}$$

Simplify to obtain:

$$R_{eq}^2 - RR_{eq} - R^2 = 0$$

Solve for R_{eq} to obtain:

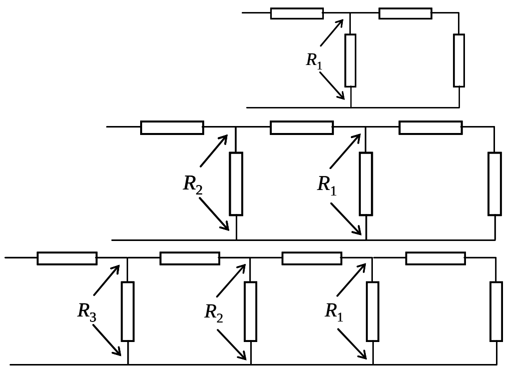
$$R_{eq} = \left(\frac{1+\sqrt{5}}{2}\right)R$$

For $R = 1\Omega$:

$$R_{eq} = \left(\frac{1+\sqrt{5}}{2}\right)(1\Omega) = \boxed{1.62\,\Omega}$$

<u>Alternative</u>:

Do it the brute force way.

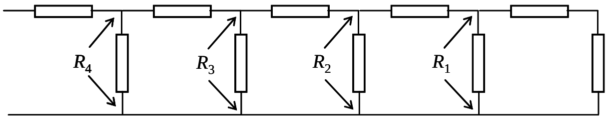


$$R_1 = \frac{1\times 2}{1+1} = 0.667\Omega.$$

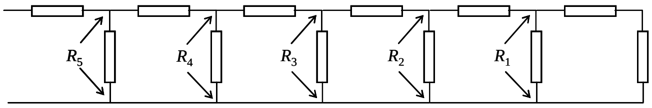
$$R_2 = \frac{1\times(1+R_1)}{1+(1+R_1)} = \frac{1\times(1+0.667)}{1+(1+0.667)} = 0.625\Omega$$

$$R_3 = \frac{1\times(1+R_2)}{1+(1+R_2)} = \frac{1\times(1+0.625)}{1+(1+0.625)} = 0.619\Omega$$

$$R_4 = \frac{1\times(1+R_3)}{1+(1+R_3)} = \frac{1\times(1+0.619)}{1+(1+0.619)} = 0.618\Omega$$



$$R_5 = \frac{1\times(1+R_4)}{1+(1+R_4)} = \frac{1\times(1+0.618)}{1+(1+0.618)} = 0.618\Omega$$



The total resistance is R = (1 + 0.618) = 1.618 Ω

Q2

(a) The total momentum of the system is $\sum_{i=2}^{N} m_i\vec{v}_c$. By momentum conservation, $\sum_{i=2}^{N} m_i\vec{v}_c + m_1\vec{v}_1 = 0$. So $\vec{v}_c = -\frac{m_1\vec{v}_1}{\sum_{i=2}^{N} m_i} = -\frac{m_1\vec{v}_1}{m_c}$, where $m_c \equiv \sum_{i=2}^{N} m_i$.

(b) The kinetic energy $KE = \frac{1}{2}\sum_{i=2}^{N} m_i(\vec{v}_c+\vec{v}_i)^2 = \frac{1}{2}\sum_{i=2}^{N} m_i v_c^2 + \frac{1}{2}\sum_{i=2}^{N} m_i v_i^2 + \sum_{i=2}^{N} m_i\vec{v}_c\cdot\vec{v}_i$. Note that $\sum_{i=2}^{N} m_i\vec{v}_i = 0$. So $\sum_{i=2}^{N} m_i\vec{v}_c\cdot\vec{v}_i = \vec{v}_c\cdot\left(\sum_{i=2}^{N} m_i\vec{v}_i\right) = 0$. So $KE = \frac{1}{2}m_c v_c^2 + \frac{1}{2}\sum_{i=2}^{N} m_i v_i^2$.

(c) Since v_c is dictated by momentum conservation, for minimum kinetic energy of the 2nd to the Nth fragments, the condition should be $v_i = 0$, i = 2, 3, …, N. Then, using the result in (a) and energy conservation $E = \frac{1}{2}m_c v_c^2 + \frac{1}{2}m_1 v_1^2$, we get

$$Max.\ KE = \frac{1}{2}m_1 v_1^2 = \frac{m_c}{m_1+m_c}E .$$

Q3

(a) Force balance $Mg + p_0(A_1 - A_2) = p(A_1 - A_2)$, so $p = p_0 + \frac{Mg}{A_1 - A_2}$

Ideal gas law: $p\left(LA_2 + (H-L)A_1\right) = NRT$

Combine the two equations, we get

$$\frac{NRT}{LA_2 + (H-L)A_1} = p_0 + \frac{Mg}{A_1 - A_2}$$

$$\frac{LA_2 + (H-L)A_1}{NRT} = \frac{A_1 - A_2}{p_0(A_1 - A_2) + Mg}$$

$$L(A_1 - A_2) = HA_1 - \frac{A_1 - A_2}{p_0(A_1 - A_2) + Mg}NRT$$

$$L = \frac{HA_1}{A_1 - A_2} - \frac{NRT}{p_0(A_1 - A_2) + Mg} .$$

(b) Take downward as positive in displacement, we have

$$F = Mg + p_0(A_1 - A_2) - p(A_1 - A_2)$$

And $p\left((L+x)A_2 + (H-L-x)A_1\right) = NRT$.

So $F = Mg + p_0\left(A_1 - A_2\right) - \dfrac{NRT\left(A_1 - A_2\right)}{\left(L+x\right)A_2 + \left(H-L-x\right)A_1}$

Tidy up the equation,

$$F = Mg + p_0\left(A_1 - A_2\right) - \frac{NRT\left(A_1 - A_2\right)}{LA_2 + \left(H-L\right)A_1 - x\left(A_1 - A_2\right)}$$

$$F = Mg + p_0\left(A_1 - A_2\right) - \frac{NRT\left(A_1 - A_2\right)}{LA_2 + \left(H-L\right)A_1}\frac{1}{1 - \dfrac{A_1 - A_2}{LA_2 + \left(H-L\right)A_1}x}$$

$$F = Mg + p_0\left(A_1 - A_2\right) - \frac{NRT\left(A_1 - A_2\right)}{LA_2 + \left(H-L\right)A_1}\left(1 + \frac{A_1 - A_2}{LA_2 + \left(H-L\right)A_1}x\right)$$

Using the answer in (a), we get

$$F = -\frac{NRT\left(A_1 - A_2\right)}{LA_2 + \left(H-L\right)A_1}\frac{A_1 - A_2}{LA_2 + \left(H-L\right)A_1}x = -NRT\left(\frac{A_1 - A_2}{LA_2 + \left(H-L\right)A_1}\right)^2 x = -kx$$

$$k = NRT\left(\frac{A_1 - A_2}{LA_2 + \left(H-L\right)A_1}\right)^2, \quad \omega = \sqrt{\frac{NRT}{M}}\frac{A_1 - A_2}{LA_2 + \left(H-L\right)A_1} = \frac{p_0\left(A_1 - A_2\right) + Mg}{\sqrt{MNRT}}$$

Q4

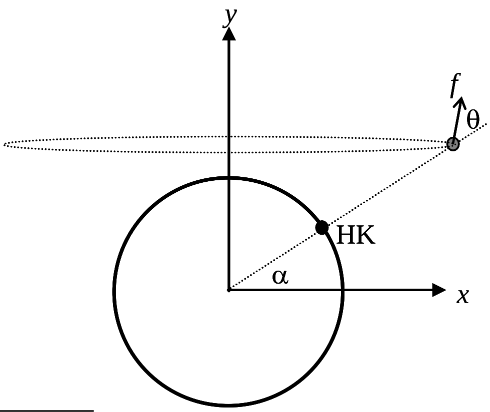
(a) $g = G\dfrac{M}{R^2}$, so $M = gR^2/G = 9.8\times(6378\times10^3)^2/(6.67\times10^{-11}) = 5.98\times10^{24}$ kg

(b) $G\dfrac{M}{r^2} = \omega^2 r = \left(\dfrac{2\pi}{T}\right)^2 r$, so

$$r = \left(G\frac{MT^2}{4\pi^2}\right)^{1/3} = \left(\frac{gR^2T^2}{4\pi^2}\right)^{1/3} = \left(\frac{9.8\times(6378\times10^3)^2\times(24\times3600)^2}{39.48}\right)^{1/3} = 4.23\times10^7 \text{ m}$$

(c)

Let the latitude be α.



$$\begin{cases} \dfrac{GM}{r^2}\sin\alpha = f\sin\left(\theta+\alpha\right) \\ \omega^2 r\cos\alpha = \dfrac{GM}{r^2}\cos\alpha - f\cos\left(\theta+\alpha\right) \end{cases}$$

Eliminate f from the above equations:

$$\cot\left(\theta+\alpha\right) = \left(1 - \frac{\omega^2 r^3}{GM}\right)\cot\alpha\ .$$

From the first equation,

$$f = \frac{GM}{r^2}\sin\alpha\csc\left(\theta+\alpha\right) = \frac{GM}{r^2}\sin\alpha\sqrt{1 + \left(1 - \frac{\omega^2 r^3}{GM}\right)^2\cot^2\alpha}$$

Hence f is minimum when

$$r=\left(\frac{\sqrt{1+8\left(1+\tan^2\alpha\right)}-1}{2\omega^2/GM}\right)^{1/3}=\left(\frac{\sqrt{1+8\sec^2\alpha}-1}{2}\right)^{1/3}\left(\frac{GM}{\omega^2}\right)^{1/3}\approx 1.034\left(\frac{GM}{\omega^2}\right)^{1/3}.$$

So the orbit radius of the satellite is about 3.4 % larger than a truly geostationary satellite above the Equator.

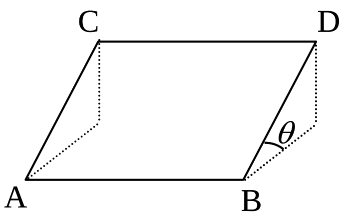
Q5

The phase difference for the matter waves to take the two paths is $\Delta\Phi=\Phi_{ABD}-\Phi_{ACD}$,

$\Phi_{ACD}=\Phi_{AC}+\Phi_{CD}$, and $\Phi_{ABD}=\Phi_{AB}+\Phi_{BD}$

Note that $\Phi_{AC}=\Phi_{BD}$,

The momentum on path AB is $p_{AB}=\sqrt{2mE_0}$, and the momentum on path CD is $p_{CD}=\sqrt{2m(E_0-mgL\sin\theta)}$.



So $\Delta\Phi=\Phi_{AB}-\Phi_{CD}=\frac{2\pi L}{h}\left(\sqrt{2mE_0}-\sqrt{2m(E_0-mgL\sin\theta}\right)$

Maximum reading of neutron number occurs when

$2n\pi=\frac{2\pi L}{h}\left(\sqrt{2m(E_0-mgL\sin\theta_n)}-\sqrt{2mE_0}\right)$, n = 0, 1, …

So the number of maximum reading is $N=\frac{L}{h}\left(\sqrt{2m(E_0-mgL)}-\sqrt{2mE_0}\right)$.