# Hong Kong Physics Olympiad 2012
# 2012 年香港物理奧林匹克競賽

## (Junior Level 初級組)

**Jointly Organized by**

**The Hong Kong Academy for Gifted Education**
香港資優教育學院
**The Education Bureau of HKSAR**
香港特區政府教育局
**The Physical Society of Hong Kong**
香港物理學會
**The Hong Kong University of Science and Technology**
香港科技大學

共同舉辦

**March 17, 2012**
**2012 年 3 月 17 日**

# Rules and Regulations 競賽規則

1. All questions are in bilingual versions. You can answer in either Chinese or English.

   所有題目均爲中英對照。你可選擇以中文或英文作答。

2. The multiple-choice answer sheet will be collected 1.5 hours after the start of the contest. You can start answering the open-ended questions any time after you have completed the multiple-choice questions without waiting for announcements.

   選擇題的答題紙將于比賽開始後一小時三十分收回。若你在這之前已完成了選擇題，你亦可開始作答開放式題目，而無須等候任何宣佈。

3. Please follow the instructions on the multiple-choice answer sheet, and use a HB pencil to write your 8-dight Participant ID Number in the field of "I. D. No.", and fill out the appropriate circles **<u>fully</u>**. After that, write your English name in the space provided and your Hong Kong ID number in the field of "Course & Section No."

   請依照選擇題答題紙的指示，用 HB 鉛筆在選擇題答題紙的 "I. D. No." 欄上首先寫上你的 8 位數字參賽號碼，並把相應寫有數字的圓圈<u>完全塗黑</u>，然後在適當的空格填上你的英文姓名，最後於 "Course & Section No. " 欄內填上你的身分證號碼。

4. After you have made the choice in answering a multiple choice question, fill the corresponding circle on the multiple-choice answer sheet **<u>fully</u>** using a HB pencil.

   選定選擇題的答案後，請將選擇題答題紙上相應的圓圈用 HB 鉛筆<u>完全塗黑</u>。

5. On the cover of the answer book, please write your Hong Kong ID number in the field of "Course Title", and write your English name in the field of "Student Name" and your 8-dight Participant I. D. Number in the field of "Student Number". You can write your answers on both sides of the sheets in the answer book.

   在答題簿封面上，請於 Course Title 欄中填上你的身分證號碼；請於 Student Name 欄中填上你的英文姓名；請於 Student Number 欄中填上你的 8 位數字參賽號碼。 答題簿可雙面使用。

6. The information provided in the text and in the figure of a question should be put to use together.

   解題時要將文字和簡圖提供的條件一起考慮。

7. Some open problems are quite long. Read the entire problem before attempting to solve them. If you cannot solve the whole problem, try to solve some parts of it. You can even use the answers in some unsolved parts as inputs to solve the others parts of a problem.

   開放題較長，最好將整題閱讀完後才著手解題。若某些部分不會做，也可把它們的答案當作已知來做其它部分。

**The following symbols and constants are used throughout the examination paper unless otherwise specified:**
**除非特別注明，否則本卷將使用下列符號和常數：**

| | | |
|---|---|---|
| Gravitational acceleration on Earth surface 地球表面重力加速度 | $g$ | 9.8 m/s$^2$ |
| Gravitational constant 萬有引力常數 | $G$ | $6.67 \times 10^{-11}$ N m$^2$/kg$^2$ |
| Radius of Earth 地球半徑 | $R_E$ | 6378 km |
| Sun-Earth distance 太陽-地球距離<br>(= 1 Astronomical Unit (AU)) (= 1 天文單位(AU)) | $r_E$ | $1.5 \times 10^{11}$ m |
| Mass of Sun 太陽質量 | $M_{\text{Sun}}$ | $1.99 \times 10^{30}$ kg |
| Mass of Earth 地球質量 | $M_E$ | $5.98 \times 10^{24}$ kg |
| Air Density 空氣密度 | $\rho_0$ | 1.2 kg/m$^3$ |
| Water Density 水密度 | $\rho_w$ | $1.0 \times 10^3$ kg/m$^3$ |
| Standard atmosphere pressure 標準大氣壓 | $p_0$ | $1.013 \times 10^5$ N/m$^2$ |
| Charge of an electron 電子電荷 | $e$ | $1.6 \times 10^{-19}$ C |
| Permittivity constant 真空電容率 | $\varepsilon_0$ | $8.85 \times 10^{-12}$ C/(V m) |
| Electron mass 電子質量 | $m_e$ | $9.11 \times 10^{-31}$ kg |
| Speed of light in vacuum 真空光速 | $c$ | $3.0 \times 10^8$ m/s |

**Trigonometric identities:**
**三角學恆等式:**

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\sin x \cos y = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]$$

$$\cos x \cos y = \frac{1}{2}\left[\cos(x+y) + \cos(x-y)\right]$$

$$\sin x \sin y = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$$

**The following conditions will be applied to all questions unless otherwise specified:**

1) All objects are near Earth surface and the gravity is pointing downwards.
2) Neglect air resistance.
3) All speeds are much smaller than the speed of light.

**除特別註明外，下列條件將適用于本卷所有問題：**

1) 所有物體都處于地球表面，重力向下；
2) 忽略空氣阻力；
3) 所有速度均遠小於光速。

# Multiple Choice Questions

(2 points each. Select one answer in each question.)

**選擇題**（每題 2 分，每題選擇一個答案。）

The MC questions with the '*' sign may require information on page-3.
帶 * 的選擇題可能需要用到第三頁上的資料。

1. As shown in Fig. 1, a boy is riding on a bus. The bus moves with a uniform speed of 20 km/h in a horizontal circle, and the traffic light is located at the center of the circle. What is the velocity of the traffic light relative to the boy?
A) 20 km/h in the forward direction of the bus
B) 20 km/h in the backward direction of the bus
C) 20 km/h perpendicular to the forward direction of the bus and directed away from the bus
D) 20 km/h perpendicular to the forward direction of the bus and directed towards the bus
E) 0 km/h

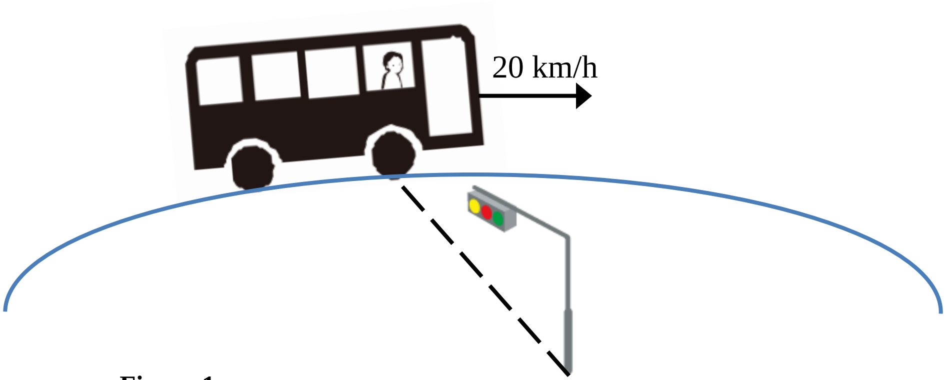



**Figure 1**

如圖 1 所示，一名男孩正在乘坐巴士，巴士以均速 20 km/h 繞著交通燈作水平圓周運動，交通燈位於圓圈的中心。交通燈相對於男孩的速度是多少？
A) 20 km/h 與巴士前進順向
B) 20 km/h 與巴士前進逆向
C) 20 km/h 與巴士前進方向垂直並遠離巴士
D) 20 km/h 與巴士前進方向垂直並趨近巴士
E) 0 km/h

Solution:
The circular motion only affects the relative acceleration. It does not affect the relative velocity.
(answer: B)

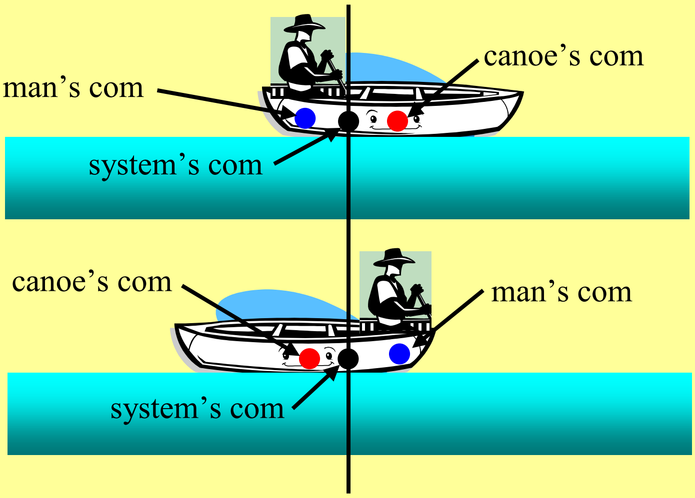
2. A tennis ball machine is installed on the bus in Fig. 1. It projects tennis balls at a speed of 100 km/h in the direction perpendicular to the forward direction of the bus, and on the outward side of the circular path. The trajectories of the tennis balls observed by the boy are
A) straight lines perpendicular to the forward direction of the bus
B) straight lines slightly inclined towards the forward direction of the bus
C) straight lines slightly inclined towards the backward direction of the bus
D) curves slightly inclined towards the forward direction of the bus
E) curves slightly inclined towards the backward direction of the bus
圖 1 的巴士裝有一台網球發射器。發射器以時速 100 km/h 沿著與巴士垂直的方向，向圓形路徑外邊發射網球。男孩觀察到的網球軌跡是
A) 與巴士前進方向垂直的直線
B) 比巴士前進方向略爲前傾的直線
C) 比巴士前進方向略爲後傾的直線

D) 比巴士前進方向略爲前傾的曲線
E) 比巴士前進方向略爲後傾的曲線

**Solution:**
Because of the curved motion path of the bus, the trajectory should be a curve. (answer: E)

3. A man sits in the back of a canoe in still water. He then moves to the front of the canoe and sits there. Neglecting the damping of water, the final position and the motion of the canoe is:
A) forward of its original position and moving forward
B) forward of its original position and moving backward
C) rearward of its original position and moving forward
D) rearward of its original position and moving backward
E) rearward of its original position and not moving
在靜止的水中，船夫坐在獨木舟的後方。他隨後向前移至獨木舟的前方然後坐下。若不計水的阻尼，獨木舟最終的位置和運動狀態是：
A) 比原來位置前移並向前動　B) 比原來位置前移並向後動
C) 比原來位置後移並向前動　D) 比原來位置後移並向後動
E) 比原來位置後移並不動

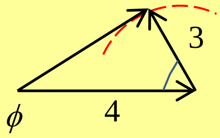
**Solution:**




Since there is no external force, the center of mass should stay at the same position. Hence the canoe is rearward of its original position and not moving. (answer: E).

4. A boat is about to cross a river to the opposite bank. The river water flows at a speed of 4 km/h. If the speed of the boat is 3 km/h, what should be the angle between the boat velocity and the upstream direction of the river, so that the downstream displacement is minimum when the boat reaches the opposite bank?
一艘船正要橫渡一道河往對岸。河水流速爲 4 km/h。若船速爲 3 km/h，那麽船速應與河的上游方向成什麽方角度，才可以在抵達對岸時，令其向下游的位移爲最小？
A) $0^o$　B) $37^o$　C) $41^o$　D) $53^o$　E) $90^o$

**Solution:**
To minimize the downstream displacement, the resultant vector of the water and boat velocity should make an angle with the downstream direction as large as possible. Considering variable directions of the boat velocity, the tip of the boat velocity generates a circle as shown in the figure. The largest angle of the resultant velocity is given by the tangent to the circle. Hence $\phi = \cos^{-1}(3/4) = 41^o$. (answer: C)




5. *In a talent show, a juggler juggles 4 balls simultaneously, as shown in Fig. 2. A spectator uses his high speed video tape and determines that it takes the juggler 0.9 s to cycle each ball through his hands (including catching, transferring and throwing) and to be ready to catch the next ball. It is noted that at most one ball must be in a hand of the juggler in each cycle of juggling. What is the minimum vertical speed the juggler must throw up each ball?
在天才表演中，一藝人同時要弄 4 個球，如圖 2 所示。有觀眾以高速攝錄機拍下過程，發現藝人需用 0.9 s 經手每個球(包括接球，傳球和拋球)，然後從新準備接下一球。他還留意到在要球的每一循環中，藝人的每隻手最多只有一個球。問藝人拋球的最低垂直速率是多少？
A) 9.3 m/s  B) 11.4 m/s  C) 12.8 m/s  D) 13.2 m/s  E) 17.6 m/s


**Figure 2**

**Solution:**

One of the balls' height can be described by $y = v_o t - \frac{1}{2}gt^2$. The amount of time it takes to rise and fall to its initial height is therefore given by $\frac{2v_o}{g}$. If the time it takes to cycle the ball through the juggler's hands is $\tau$ = 0.9 s, then there must be 3 balls in the air during that time $\tau$. A single ball must stay in the air for at least $3\tau$ so the condition is $\frac{2v_o}{g} \geq 3\tau$, or $v_o \geq 13.2$ m/s. (answer: D)

6. As shown in Fig. 3, the block-spring system is in equilibrium provided that the left spring is stretched by $x_1$. The whole system rests on a smooth supporting surface. The coefficient of static friction between the blocks is $\mu_s$, and the blocks have equal mass $m$. What is the maximum amplitude of the oscillations of the system such that the top block does not slide on the bottom one?

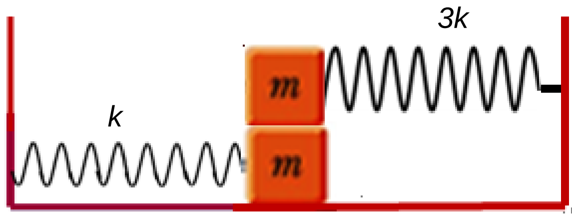


**Figure 3**

如圖 2 所示，一個方塊-彈簧系統處在平衡態中，其中左彈簧伸展長度爲 $x_1$。全系統安放在光滑面上。方塊之間的靜摩擦係數爲 $\mu_s$，方塊的質量均爲 $m$。問系統振動可達到的最大幅度，而上下方塊之間仍能保持不滑動。

A) $\frac{\mu_s mg}{2k} + x_1$  B) $4kx_1 - \mu_s mg$  C) $\frac{\mu_s mg}{k} - x_1$  D) $2(2kx_1 + \mu_s mg)$  E) $2(2kx_1 - \mu_s mg)$

**Solution:**
The equivalent force constant of the springs in the system is $(3k+k) = 4k$
Applying Newton's 2nd law to the two-block system gives: $-4kx = 2ma$  (1)
Applying Newton's 2nd law to the lower block gives: $k(x_1 - x) - f = ma$,  (2)
where $f$ is the magnitude of the frictional force.

Solving the Eq. (1) for $ma$ and substituting the result into Eq. (2) gives
$k(x_1 - x) - f = -2kx$.

Solving for $f$: $f = k(x_1 + x)$.

The maximum value for $x$ is the amplitude $A$, and the maximum value for $f$ is $\mu_s N = (\mu_s)mg$. Thus, $(\mu_s)mg = k(x_1 + A_{max})$. Solving for $A_{max}$ gives

$$A_{max} = \frac{\mu_s mg}{k} - x_1 \quad \text{(answer: C)}$$

7. *An object is projected up an inclined plane with an initial speed of $v_0$ = 10 m/s, as shown in Fig. 4. The angle of the incline is $\theta$ = 30° above the horizontal direction and the coefficient of the sliding friction $\mu_k$ = 0.1, determine the total time for the object to return to the point of projection.
一物體沿斜面以初速 $v_0$ = 10 m/s 射出，如圖 4 所示。斜面的水平傾角為 $\theta$ = 30°，動摩擦係數為 $\mu_k$ = 0.1，求物體回到原點的總時間。
A) 3.81 s B) 4.26 s C) 4.54 s D) 4.94 s E) 5.32 s

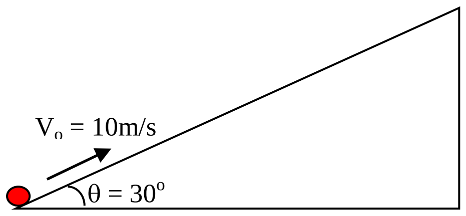



**Figure 4**

**Solution:**

Going up: $F_x = -mg\sin 30^\circ - \mu mg\cos 30^\circ = ma$

$$\therefore a = -g[\sin 30^o + (0.1)\cos 30^o] = -5.75\,\text{m/s}^2$$

At the highest point $v = 0$ so $v = v_\circ + at \Rightarrow t_{up} = -\frac{v_o}{a} = -\frac{v_o}{-(5.75)} = 0.174 v_o$ s

The length of the inclined plane is given by

$$S_{up} = v_o t_{iu} + \frac{1}{2}at_{up}^2 = 0.174 v_o^2 - \frac{1}{2}(5.75)(0.174 v_o)^2 = 0.087 v_o^2$$

Going down: $v_\circ{}' = 0$, $a = -g[\sin 30^o - (0.1)\cos 30^o] = -9.8(0.5 - 0.0866) = 4.05\,\text{m/s}^2$

$$S_{down} = v_o' t_{down} + \frac{1}{2}at_{up}^2 = 0 - \frac{1}{2}(4.05)(t_{down}^2)$$

$$0.087 v_o^2 = \frac{1}{2}(4.05)(t_{down}^2)$$

$$t_{down} = 0.207 v_\circ\ s$$

$$\therefore t_{total} = t_{up} + t_{down} = 0.381 v_o = 0.381(10) = 3.81\,\text{s} \quad \text{(answer: A)}$$

8. *As shown Fig. 5, a train with a length of $L$ = 500 m moves by its inertia through the horizontal section of a railroad. However, the train encounters a small hill that slopes gently. With what minimum speed $v$ can the train cross the hill? The base of hill has a length $\ell$ = 100 m, the lengths of the slopes are $\ell_1$ = 80 m and $\ell_2$ = 60 m. The slopes of hill can be considered as straight lines and the small section of rounding at the top of the hill can be ignored. Neglect any friction.
如圖 5 所示，一列長度為 $L$ = 500 m 的火車，以慣性在水平軌道上滑行，然後遇到一個小山丘。問火車需以什麼最低速度 $v$ 越過山丘？山丘底長 $l$ = 100 m，兩邊斜坡長 $l_1$ = 80 m 和 $l_2$ = 60 m。可以考慮斜坡為直線，坡頂彎曲的一小段可以忽略。可以忽略摩擦力。
A) 9.6 m/s B) 11.5 m/s C) 13.2 m/s D) 15.0 m/s E) 16.2 m/s



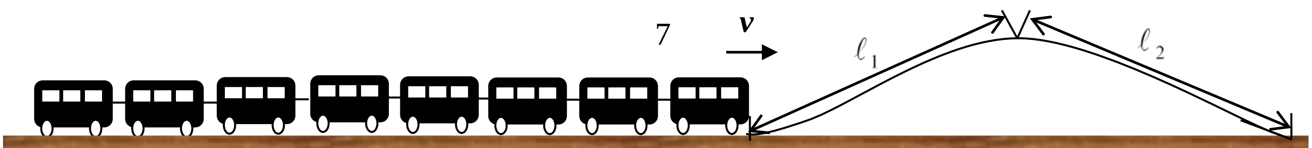


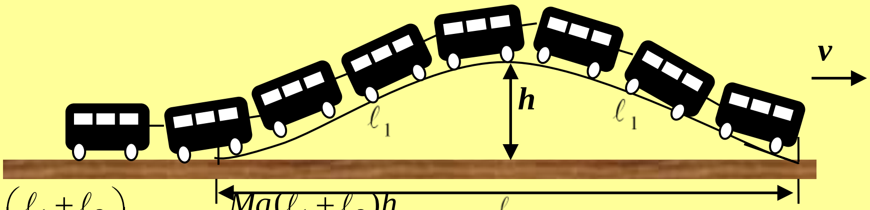
**Figure 5**

**Solution:**
The potential energy of the train is maximum when parts of the train completely occupy both slopes of hill.
Using cosine rule, the inclination of the left slope is given by

$$\cos\theta = \frac{l^2 + l_1^2 - l_2^2}{2ll_1} = 0.8$$

$$\sin\theta = \sqrt{1-\cos^2\theta} = 0.6$$

Height of the hill: $h = l_1 \sin\theta = 48\,\text{m}$
Alternatively, as can be seen from the given conditions, the lengths of slopes and base of hill satisfy the Pythagorean theorem: $\sqrt{\ell_1^2 + \ell_2^2} = \ell$. Hence it is a right-angled triangle. By equating the area of the triangle, the height of the hill $h = \ell_1 \ell_2 / \ell$ = 48 m.
The center of mass of the lifted part of the train, which is located on the hill, gives $h_{c.g.} = h/2$. Since the mass of the parts of the train indicated is proportional to its length, it is possible to find potential energy of train relative to the foot of the hill:




$$P.E. = M\left(\frac{\ell_1 + \ell_2}{L}\right) g h_{c.g.} = \frac{Mg(\ell_1 + \ell_2)h}{2L}$$ ; where $M$ is the mass of the entire train.

If the train can cross the hill, its initial kinetic energy $K.E. = Mv^2/2$ must be larger than the potential energy $U$. Hence the desired minimum speed of the train is

$$v_{\min} = \sqrt{gh\left(\frac{\ell_1 + \ell_2}{L}\right)} \approx 11.5\,\text{m/s}$$ (answer: B)

9. It is given the mass of Earth is $a$ times that of Moon, and the radius of Earth is $b$ times that of Moon. The period of a simple pendulum is $T$. When it is carried to Moon, the period of the simple pendulum becomes
已知地球的質量是月球的 $a$ 倍，地球半徑是月球的 $b$ 倍。現有一個單擺，在地球上的週期爲 $T$，若將它移到月球上，則單擺的週期等於

A) $\frac{a}{\sqrt{b}}T$ B) $\frac{\sqrt{b}}{a}T$ C) $\frac{\sqrt{a}}{b}T$ D) $\frac{b}{\sqrt{a}}T$ E) $\sqrt{\frac{b}{a}}T$

Solution:

$$mg = G\frac{mM}{R^2},\ g = \frac{GM}{R^2},\ T = 2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{l}{G}}\frac{R}{\sqrt{M}},\ \frac{T_2}{T_1} = \frac{R_2}{R_1}\sqrt{\frac{M_1}{M_2}} = \frac{\sqrt{a}}{b}.$$ (answer: C)

10. There exist some triple star systems in the universe. They are more distant from other stars, and are composed of three stars of equal mass $M$. The gravitational forces due to other stars can be

neglected. A basic stable structure of triple star systems consists of three collinear stars, with two companion stars moving around a central star on a circular orbit with radius $R$. The linear velocity of the companion stars is $v_1 = k_1\sqrt{GM/R}$, where $k_1$ =

宇宙中存在一些離其他恒星較遠的、由質量 $M$ 相等的三顆星組成的三星系統，通常可忽略其他星體對它們的引力作用。已觀測到穩定的三星系統存在的一種基本構成形式，是三顆星位於同一直線上，兩顆伴星圍繞中央星在半徑爲 $R$ 的圓軌道上運行。伴星運動的線速度 $v_1 = k_1\sqrt{GM/R}$，式中 $k_1$ =

A) $\sqrt{10}$ B) $\sqrt{5}$ C) $\sqrt{5/2}$ D) $\sqrt{5}/2$ E) $\sqrt[3]{5/2}$

**Solution**

$\Sigma F = F_1 + F_2 = \frac{Gm^2}{R^2} + \frac{Gm^2}{(2R)^2} = \frac{5Gm^2}{4R^2} = mR\omega^2$, $\omega = \sqrt{\frac{5Gm}{4R^3}}$, $v_1 = R\omega = \frac{\sqrt{5}}{2}\sqrt{\frac{Gm}{R}}$. (answer: D)

## 《END OF MC's 選擇題完》

# Open Problems 開放題

## Total 6 problems 共 6 題

The Open Problem(s) with the '*' sign may require information on page 3.
帶 * 的開放題可能需要用到第三頁上的資料。

**Q1* (10 points)** **題1*（10分）**

The helicopter has a mass $m$ and maintains its height by imparting a downward momentum to a column of air defined by the slipstream boundary as shown in Fig. 6. The propeller blades can project a downward air speed $v$, where the pressure in the stream below the blades is atmospheric and the radius of the circular cross-section of the slipstream is $r$. Neglect any rotational energy of the air, the temperature rise due to air friction and any change in air density $\rho$.

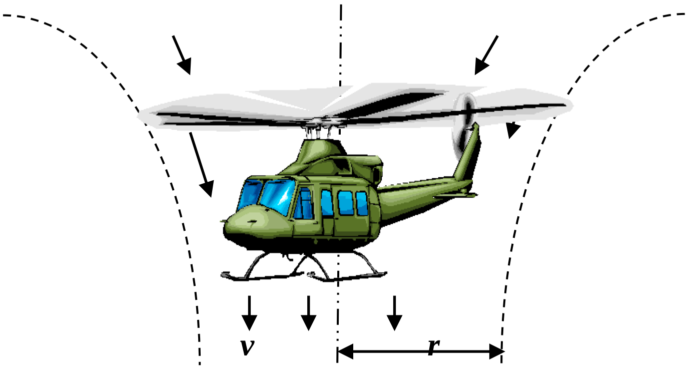


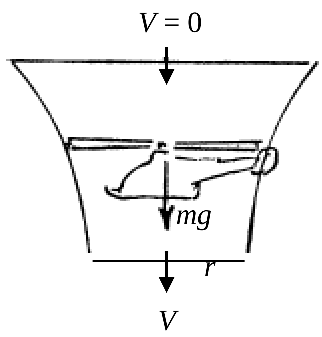
**Figure 6**

(a) Determine the power $P$ required of the engine.
(b) If the power is doubled, calculate the acceleration of the helicopter.

直升機的質量爲 $m$，它傳輸下向的動量予空氣柱，用以維持其高度。氣流的邊界如圖 6 所示。它的引擎業片能鼓動空氣向下流，速度達到 $v$。氣流中的壓強爲大氣壓強，氣流的切面爲圓形，半徑爲 $r$。可以忽略空氣的轉動能、空氣摩擦引起的升溫、和空氣密度的改變。
(a) 試推導引擎所需的功率 $P$。
(b) 如果把功率增至兩倍，試計算直升機的加速度。

**Solution:**

(a) Using Newton's second law, the force exerted by the engine is equal to the rate of change of momentum of the air.




Mass of air propelled by the enginer in time $t = \rho vAt$

Increase in momentum of the air in time $t = (\rho vAt)v = \rho v^2 At$.

Hence the force is $F = \dfrac{\rho v^2 At}{t} = \rho v^2 A = \rho\pi r^2 v^2$.

For the helicopter to maintain its height, $F = mg$.

Hence $mg = \rho\pi r^2 v^2 \quad \Rightarrow \quad v = \dfrac{1}{r}\sqrt{\dfrac{mg}{\pi\rho}}$.

The increase of kinetic energy of the air in time $t$ $K = \dfrac{1}{2}(\rho vAt)v^2$

Hence the power is $P = \dfrac{K}{t} = \dfrac{1}{2}\rho\pi r^2 v^3 = \dfrac{mg}{2r}\sqrt{\dfrac{mg}{\pi\rho}}$. (answer)

(b) Since $P$ is proportional to $v^3$, the air velocity becomes $\sqrt[3]{2}v$ when the power is doubled.

Since $F$ is proportional to $v^2$, the force becomes $\sqrt[3]{4}F = \sqrt[3]{4}mg$ when the power is doubled.

Using Newton's second law, the acceleration of the helicopter is given by

$\sqrt[3]{4}mg - mg = ma \quad \Rightarrow \quad a = \left(\sqrt[3]{4} - 1\right)g = 5.76\,\text{m/s}^2$ (answer)

**Q2* (10 points)** **題 2*（10 分）**

As shown in Fig. 7, in a ski jumping competition on a slope with inclination angle $\theta$ = $60^0$, an althlete jumps at point O with initial speed $v_0$ = 25 m/sand lands at point-A.
(a) Find the optimum jumping angle $\alpha$ so that the distance OA is maximum. (You may need to use trigonometric identities listed in page 3.)
(b) Find the maximum distance OA.

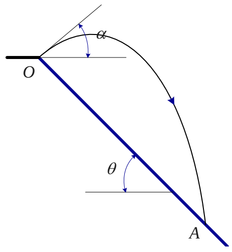
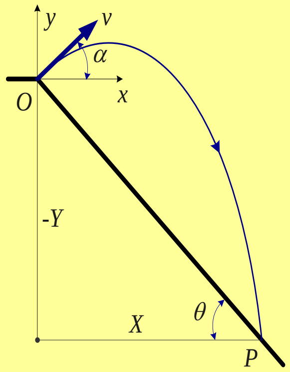


**Figure 7**

如圖 7 所示，在傾角$\theta$ = $60^0$的雪坡上舉行跳台滑雪比賽。運動員在起跳點 $O$ 以速率 $v_0$ = 25 m/s、仰角$\alpha$起跳，最後落在斜坡上 $A$ 點。取 $O$、$A$ 兩點的距離 $L$ 爲比賽成绩。
(a)求運動員可以跳得最遠距離 OA 的起跳角$\alpha$。(你或許會用到第 3 頁的三角學恆等式。)
(b) 求最遠距離 OA。

**Solution:**




Set the coordinate axes as shown in the figure. At time T,

$$X = vT\cos\alpha\,, \qquad (1)$$

$$Y = vT\sin\alpha - \frac{1}{2}gT^2\,. \qquad (2)$$

They satisfy the equation $Y = -X\tan\theta\,.$ (3)

From Eqs. (1) to (3), $gT^2 - 2vT(\sin\alpha + \cos\alpha\tan\theta) = 0\,.$

We ontain $T = \dfrac{2v}{g}(\sin\alpha + \cos\alpha\tan\theta) = \dfrac{2v}{g}\dfrac{\sin(\alpha+\theta)}{\cos\theta}$ (4)

Substituting Eq. (4) into Eq. (1), $X = \dfrac{2v^2}{g}\dfrac{\sin(\theta+\alpha)\cos\alpha}{\cos\theta}$ (5)

The distance $OP$ along the slope is given by $L(\alpha) = \dfrac{X}{\cos\theta} = \dfrac{2v^2}{g}\dfrac{\sin(\theta+\alpha)\cos\alpha}{\cos^2\theta}$. (6)

To calculate the best inclination and the maximum distance, we need to maximize $L$ with respect to $\alpha$.

From the trigonometric identity: $\sin x\cos y = \dfrac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]$ , we find

$$\sin(\theta+\alpha)\cos\alpha = \frac{\sin(\theta+2\alpha) + \sin\theta}{2}\,.$$

Hence $L(\alpha) = \dfrac{v^2}{g}\dfrac{\sin(\theta+2\alpha)+\sin\theta}{\cos^2\theta}$. When $\sin(\theta+2\alpha)$ =1, that is, $\theta$+2$\alpha$ =$90^0$, $\alpha_0 = \dfrac{90^0 - \theta}{2}$ =$15^o$,

and the maximum distance is $L_{\max} = L(\alpha_0) = \dfrac{v^2}{g}\dfrac{1+\sin\theta}{\cos^2\theta} = \dfrac{25^2}{9.8}\dfrac{1+\sin 60^0}{\cos^2 60^0} = 255.1\left(1+\dfrac{\sqrt{3}}{2}\right)$.

The optimal jumping angle $\alpha$ is $15^0$, and the maximum distance is 476 m. (answer)

An alternative solution can be obtained by setting the slope direction as the X axis, and the perpendicular direction as the Y axis. Then

$x = v\cos(\theta+\alpha)t + \frac{1}{2}(g\sin\theta)t^2$ (a)

$y = v\sin(\theta+\alpha)t - \frac{1}{2}(g\cos\theta)t^2$ (b)

When $y = 0$, $t = \frac{2v\sin(\theta+\alpha)}{g\cos\theta}$. Substituting into (a),

$$x = \frac{2v^2\sin(\theta+\alpha)}{g\cos^2\theta}\left[\cos(\theta+\alpha)\cos\theta + \sin\theta\sin(\theta+\alpha)\right].$$

Using the trigonometric identity $\cos(x-y) = \cos x\cos y + \sin x\sin y$, we have

$$x = \frac{2v^2\sin(\theta+\alpha)\cos\alpha}{g\cos^2\theta}.$$

Using the trigonometric identity $\sin x\cos y = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]$,

$x = \frac{v^2}{g\cos^2\theta}\left[\sin(\theta+2\alpha) + \sin\theta\right]$. When $\sin(\theta+2\alpha)=1$, that is, $\theta+2\alpha=90^0$, $\alpha_0 = \frac{90^0-\theta}{2}=15^o$

and the maximum distance is $L_{max} = L(\alpha_0) = \frac{v^2}{g}\frac{1+\sin\theta}{\cos^2\theta} = \frac{25^2}{9.8}\frac{1+\sin 60^0}{\cos^2 60^0} = 255.1\left(1+\frac{\sqrt{3}}{2}\right)$.

The optimal jumping angle $\alpha$ is $15^0$, and the maximum distance is 476 m. (answer)

**Q3* (15 points) 題 3*（15 分）**

On a smooth horizontal surface shown in Fig. 8, there is a rectangular board *AB* with mass 2*m* and an arc-shaped structure *BC* with mass 3*m*. The coefficient of kinetic friction of the upper surface of the rectangular board is $\mu$ = 0.35, and the surface of the arc-shaped structure *BC* is smooth. The arc *BC* subtends a right angle with radius *R*. The two objects touch at point *B*.

(a) A small object of mass *m* moves from *A* to the right with an initial velocity $v_0$ = 10 m/s on the upper surface of the rectangular block. When it reaches point *B*, the velocity is *v* = 5 m/s. Calculate the velocity $v_L$ of the rectangular board *AB* at this instant.

(b) The small object continues to move onto the arc-shaped surface *BC*. The rectangular board *AB* loses contact with the arc-shaped structure *BC*, and the small object continues to to move along the arc surface, and finally just reaches the highest point *C* of the arc *BC*. Calculate the velocity of the arc-shaped structure $v_R$.

(c) Calculate the length *L* and the radius *R*.

質量爲 2*m* 的長方體 *AB* 和質量爲 3*m* 的圓弧體 *BC* 靜止在圖 8 中的光滑水平面上，長方體上表面的動摩擦系數爲$\mu$ = 0.35，圓弧體 *BC* 表面則爲光滑圓弧。圓弧 *BC* 的夾角爲直角，半徑爲 *R*。兩物體於 *B* 點接觸。

(a) 質量爲 *m* 的小滑塊在 *A* 點以初速度 $v_0$ = 10 m/s 沿長方體方板上表面向右滑動，滑過 *B* 點的速率爲 *v* = 5m/s。試求此刻長方體 *AB* 的速率 $v_L$。

(b) 滑塊然後滑到圓弧體表面 *BC* 上，長方體 *AB* 和圓弧體 *BC* 失去接觸，滑塊則繼續在圓弧表面上滑動，最終恰好能滑到圓弧 *BC* 上的最高點 *C*。試求此刻圓弧體 *BC* 的速率 $v_R$。

(c) 試求長度 *L* 和圓弧半徑 *R*。

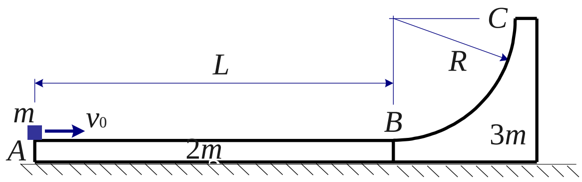



**Figure 8**

**Solution:**

$v_0$=10m/s, $v$=5m/s, $\mu$=0.35

(a) Momentum conservation from $A$ to $B$, $mv_0 = mv + (m_1 + m_2)v_L \Rightarrow v_L = \dfrac{m(v_0 - v)}{m_1 + m_2} = \dfrac{v_0}{10}$ =1 m/s (answer)

(b) Momentum conservation from $B$ to $C$,

$$mv + m_2 v_L = (m + m_2)v_R,\ m(\frac{v_0}{2}) + 3m(\frac{v_0}{10}) = (m + 3m)v_R \Rightarrow$$

$$v_R = \frac{mv + m_2 v_L}{m + m_2} = \frac{1}{m + m_2}\left(mv + m_2\frac{m(v_0 - v)}{m_1 + m_2}\right) = \frac{m(m_1 v + m_2 v_0)}{(m + m_2)(m_1 + m_2)} = \frac{v_0}{5} = 2\text{ m/s} \quad \text{(answer)}$$

(c) Energy conservation from $A$ to $B$, $\dfrac{1}{2}mv_0^2 = \dfrac{1}{2}mv^2 + \dfrac{1}{2}(m_1 + m_2)v_L^2 + \mu mgL \Rightarrow$

$$L = \frac{m(v_0^2 - v^2) - (m_1 + m_2)v_L^2}{2\mu mg} = \frac{(m_1 + m_2)(v_0^2 - v^2) - m(v_0 - v)^2}{2\mu g(m_1 + m_2)} = \frac{7}{20}\frac{v_0^2}{\mu g} = 10.2\text{ m} \quad \text{(answer)}$$

Energy conservation from $B$ to $C$, $\dfrac{1}{2}mv^2 + \dfrac{1}{2}m_2 v_L^2 = \dfrac{1}{2}(m + m_2)v_R^2 + mgR$,

$$m(\frac{v_0}{2})^2 + 3m(\frac{v_0}{10})^2 = (m + 3m)(\frac{v_0}{5})^2 + 2mgR \Rightarrow$$

$$R = \frac{mv^2 + m_2 v_L^2 - (m + m_2)v_R^2}{2mg} = \frac{3}{50}\frac{v_0^2}{g} = 0.61\text{ m} \quad \text{(answer)}$$

**Q4* (10 points)　　題 4*（10 分）**

Geosynchronous satellites have the same period $T$ as the Earth's rotation. They are at such a height above the earth's surface $h$ that they remain always above the same spot. Suppose a geosynchronous solar satellite, as shown in Fig. 9a, sends radio signal directly to receivers on earth.


**Figure 9a**

When the satellite moves to the rear part of the earth (Fig. 9b), the light source is completely blocked by Earth in the shadow region (commonly known as an umbra). The length of the umbra is usually $n$ times ($n \approx 200$) of Earth's radius $R_E$.

(a) Determine the height $h$. Express your answer in units of $R_E$.

(b) Determine the duration in each day that the satellite cannot receive sunlight. Express your answer in minutes.

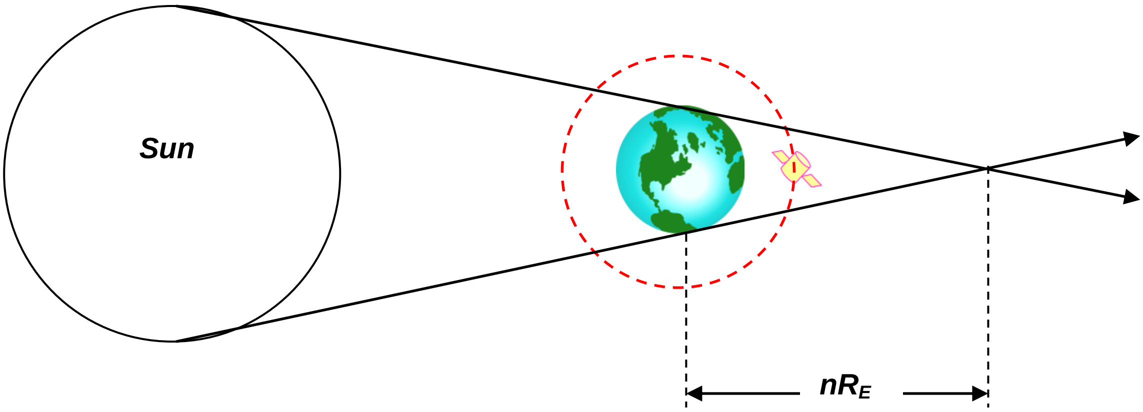



**Figure 9b**

地球同步衛星的軌道周期 $T$，與地球自轉週期相同。它們處於離地面高度 $h$ 的位置，使它們經常位於同一地點上空。如圖 9a 所示，設有一地球同步的太陽能衛星，直接傳送無線電信號給地球上的接收站。

當衛星航行到地球背面(圖 9b)，光源被地球完全遮擋形成全影區。全影區的長度通常爲地球半徑 $R_E$ 的 $n$ 倍($n \approx 200$)。

(a) 試計算高度 $h$，以 $R_E$ 爲單位表達結果。

(b) 試計算衛星每日不能接收陽光的時間，以分鐘爲單位表達結果。

**Solution:**

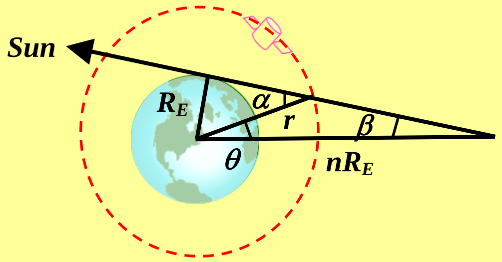
(a) Let $r$ be the distance of the satellite measured from the centre of the earth. When the satellite moves in the circular orbit, its centripetal force is

$$G\frac{mM_e}{r^2} = ma = m(\frac{4\pi^2}{T^2})r; \quad \Rightarrow r = \sqrt[3]{\frac{GM_eT^2}{4\pi^2}} = \sqrt[3]{\frac{(6.67\times10^{-11})(5.98\times10^{24})(86400)^2}{4\pi^2}} = 4.225\times10^7 \text{ m}$$

$$r = \frac{4.225\times10^7}{6.378\times10^6} = 6.62\,R_E$$

Hence $h = 5.62\ R_E$. (answer)

(b)




As shown in the figure,

$$\sin\alpha = \frac{R_E}{r} = \frac{1}{6.6244} \quad \Rightarrow \quad \alpha = 0.1515\text{rad}$$

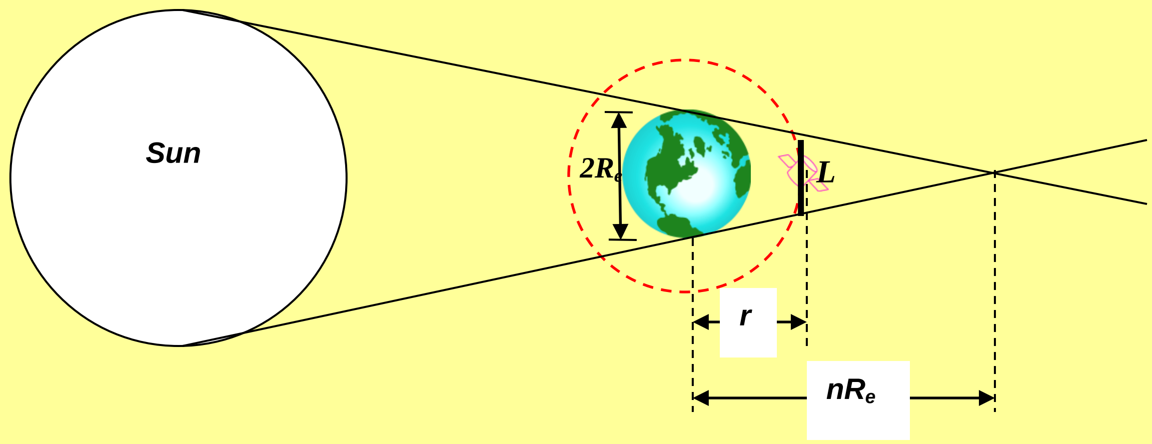
$$\sin\beta = \frac{R_E}{nR_E} = \frac{1}{200} \quad \Rightarrow \quad \beta = 0.005\text{rad}$$

$$\theta = \alpha - \beta = 0.1515 - 0.005 = 0.1465\text{rad}$$

Hence the length of the black-out period is

$$t = (24)(60)\frac{(2)(0.1465)}{2\pi} = 67.2\text{ min} \quad \text{(answer)}$$

An alternative, but approximate, solution can be obtained by assuming $nR_e$ is very large compared with $L$, so that the orbit in the region of umbra can be treated as a straight line $L$.




By using simple trigonometry, we have the following equations:

$$\frac{L}{2R_e} = \frac{nR_e - r}{nR_e}, \Rightarrow L = 2R_e - \frac{2r}{n}$$

The time the satellite moves in the region of umbra is

$$t = (\frac{L}{2\pi r})T = \left(\frac{R_e}{\pi r} - \frac{1}{n\pi}\right) \cdot T \quad \ldots(2)$$

Substitute equation (1) into (2) which yields

$$t = \left(\sqrt[3]{\frac{4R_e}{\pi g T^2}} - \frac{1}{n\pi}\right) \cdot T$$ , where $T = 24 \times 3600 = 86400$ s, $R_e = 6{,}378{,}000$ m

$$= \left(\sqrt[3]{\frac{4(6378000)}{(3.14)(9.8)(86400)^2}} - \frac{1}{(200)(3.14)}\right) \cdot (86400)$$

$$= (0.048 - 1.59 \times 10^{-3}) \cdot (86400) = 4010s = 66.8\,\text{min}$$

**Q5* (20 points)　　題 5*（20 分）**

Let $r_E$ = 1.00 AU be the circular orbtal radius of Earth around Sun, and $r_V$ = 0.72 AU be the circular orbital radius of Venus around Sun. To launch a space probe from Earth to Venus, the space probe first enters Earth's orbit and moves to a position sufficiently remote from Earth, so that the gravitational attraction of Earth is negligible when compared with the gravitational attraction of Sun. Then, the following manouvers as shown in Fig. 10 are performed:

(1) The kinetic energy of the space probe is reduced by $\Delta K = K(r_E - r_V)/(r_E + r_V)$, where $K$ is the kinetic energy of the space probe at that instant. This is done by switching on the engine for a short time and then switching it off. The space probe then enters a transfer orbit around the Sun. The transfer orbit is tangential to Earth's orbit at its near end and to Venus's orbit at its far end.

(2) When the space probe touches Venus's orbit, the space probe is traveling too fast for it to land on Venus. For the second time, the kinetic energy is reduced by switching on the engine for a short time and then switching it off. The space probe then travels at the orbital velocity of Venus.

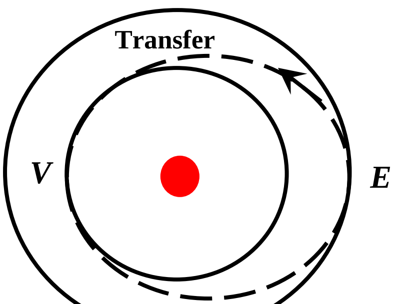



**Figure 10**

(a) Calculate the orbital speed of Earth and Venus. Express your answer in km/s.
(b) Calculate the speed (in km/s) of the space probe in the transfer orbit at point *E*.
(c) Calculate the speed (in km/s) of the space probe in the transfer orbit at point *V*.
(d) Calculate the fractional reduction of the kinetic energy of the space probe at point *V*.

設 $r_E$ = 1.00 AU 爲地球環繞太陽圓形軌道的半徑，$r_V$ = 0.72 AU 爲金星環繞太陽圓形軌道的半徑。發射一個太空探測器從地球到金星，可以先將探測器放進地球軌跡，令它處於離地球較遠的位置，以致地球的引力相對於太陽引力，可以忽略不計。然後對探測器進行如圖 10 的操作：

(1) 把探測器的動能降低 $\Delta K = K(r_E - r_V)/(r_E + r_V)$，其中 $K$ 爲探測器在該時刻的動能，方法是開動引擎一段短時間，然後關掉引擎。探測器因而進入環繞太陽的轉移軌道。這轉移軌道在近端與地球軌道相切，在遠端則與金星軌道相切。

(2) 當探測器切入金星軌道，它的高速令它難以降落金星表面。它需要第二次開動引擎一段短時間，然後關掉引擎。探測器因而降低動能，以金星軌道速率航行。

(a) 試計算地球和金星在軌道上的速率，以 km/s 爲單位表達結果。
(b) 試計算探測器在轉移軌道 $E$ 點的速率，以 km/s 爲單位表達結果。
(c) 試計算探測器在轉移軌道 $V$ 點的速率，以 km/s 爲單位表達結果。
(d) 探測器在轉移軌道 $V$ 點時，需降低的動能爲該時刻動能的什麼比例?

**Solution:**

(a) Using Newton's second law,

$$M_E \frac{v_E^2}{r_E} = \frac{GM_{Sun}M_E}{r_E^2} \quad \Rightarrow \quad v_E = \sqrt{\frac{GM_{Sun}}{r_E}} = \sqrt{\frac{(6.67\times10^{-11})(1.99\times10^{30})}{1.5\times10^{11}}} = 29747\text{ m/s} \approx 29.75\text{ km/s}$$

(answer)

Similarly, $v_V = \sqrt{\frac{GM_{Sun}}{r_V}} = \sqrt{\frac{(6.67\times10^{-11})(1.99\times10^{30})}{(0.72)(1.5\times10^{11})}} = 35057\text{ m/s} \approx 35.06\text{ km/s}$ (answer)

(b) At point *E* of Earth's orbit, kinetic energy: $K = \frac{1}{2}mv_E^2 = \frac{GM_{Sun}m}{2r_E}$.

At point *E* of the transfer orbit, kinetic energy:

$$\frac{1}{2}mv_1^2 = \frac{1}{2}mv_E^2\left(1 - \frac{r_E - r_V}{r_E + r_V}\right) \quad \Rightarrow \quad v_1 = v_E\sqrt{\frac{2r_V}{r_E + r_V}} = 27218\text{ m/s} \approx 27.22\text{ km/s} \quad \text{(answer)}$$

(c) Using conservation of energy from point *E* to point *V*,

$$\frac{1}{2}mv_1^2 - \frac{GM_{Sun}m}{r_E} = \frac{1}{2}mv_2^2 - \frac{GM_{Sun}m}{r_V}$$

$$v_2 = \sqrt{v_1^2 - \frac{2GM_{Sun}}{r_E} + \frac{2GM_{Sun}}{r_V}} = \sqrt{\frac{2GM_{Sun}r_E}{r_V(r_E + r_V)}} = v_V\sqrt{\frac{2r_E}{r_E + r_V}} = 37803\text{ m/s} \approx 37.80\text{ km/s} \quad \text{(answer)}$$

(d) Fractional rduction of kinetic energy:

$$\frac{\Delta K_2}{K_2} = \frac{\frac{1}{2}mv_2^2 - \frac{1}{2}mv_V^2}{\frac{1}{2}mv_2^2} = \frac{v_2^2 - v_V^2}{v_2^2} = \frac{\frac{2r_E}{r_E + r_V} - 1}{\frac{2r_E}{r_E + r_V}} = \frac{r_E - r_V}{2r_E} = \frac{1 - 0.72}{2} = 0.14 \quad \text{(answer)}$$

**Q6 (15 points)** **題 6（15 分）**

As shown in Fig. 11, a pendulum consists of a massive cubic block with side length $b$ and density $\rho_b$ hung by a light rigid rod with length $L >> b$. The hinge connecting the rod and the block maintains the block at the same orientation when it swings. The block is partially immersed in a liquid with density $\rho_l$, with $\rho_l$ greater than $\rho_b$. The immersion depth of the block in the liquid is $c$.

(a) Calculate the period of the pendulum for small oscillation angles $\phi$ when $c$ is sufficiently small. You may assume that the damping of the liquid during oscillations is negligible, and the change of the immersion depth during oscillations is negligible.

(b) When liquid is added, the liquid level rises, and the value of $c$ reaches a threshold value $c^*$, above which the pendulum rod is no longer vertical. What is the espression of $c^*$?

(c) When $c$ is above $c^*$, the pendulum oscillates about a non-zero angle $\phi$. Calculate the period of the pendulum for small oscillations when the liquid level has risen to one that corresponds to $\phi = 30^o$.

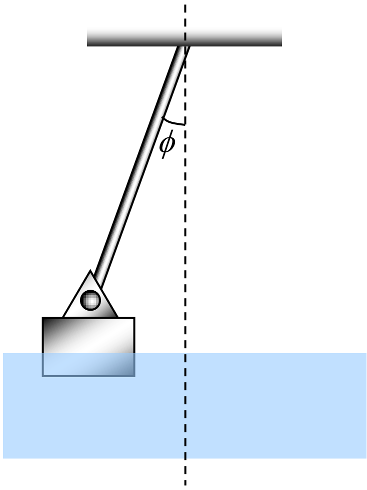


**Figure 11**

如圖 11 所示，在一個鐘擺中，一根無質量的剛體長桿懸掛著一個有質量的立方體。立方體邊長爲 $b$，密度爲 $\rho_b$。長桿長度爲 $L >> b$。連接長桿和立方體的樞紐，在鐘擺運動時，把立方體維持在同一方向。立方體部分浸在液體中，液體密度爲 $\rho_l$，數值比 $\rho_b$ 大。立方體浸在液體中的深度爲 $c$。

(a) 試計算當 $c$ 充分小的時候，鐘擺作小角度振動的周期。可以假設在振動時，液體的阻尼可以忽略，也可以假設在振動時，立方體浸在液體中的深度改變可以忽略。

(b) 當液體增加時，液體表面上升，$c$ 的數值達到閾值 $c^*$。在閾值以上，鐘擺的長桿不再垂直。$c^*$的表達式爲何？

(c) 當 $c$ 在 $c^*$以上時，鐘擺環繞非零值的$\phi$振動。試計算當液面升至$\phi = 30^o$的位置時，鐘擺作小角度振動的周期。

Solution:

(a) Weight of the block: $W = \rho_b b^3 g$

Buoyancy of the block: $B = \rho_l b^2 cg$

Hence $W - B = \rho_b b^3 g - \rho_l b^2 cg = \rho_b b^3 g\left(1 - \frac{\rho_l c}{\rho_b b}\right)$.

This implies that the pendulum is equivalent to one that is placed in a gravitational field with gravitational acceleration $g\left(1 - \frac{\rho_l c}{\rho_b b}\right)$. Hence the period of the pendulum is $T = 2\pi\sqrt{\frac{L}{g\left(1 - \frac{\rho_l c}{\rho_b b}\right)}}$.

(answer)

(b) When $c$ approaches $\frac{\rho_b b}{\rho_l}$, then we have $B$ approaches $W$. Above this value, the block will float, and the pendulum is no longer vertical. Hence $c^* = \frac{\rho_b b}{\rho_l}$. (answer)

(c) Let $x$ be the displacement of the block along the tangential direction of the rigid rod swing.
$x$ is positive in the direction of increasing $\phi$.
Then the vertical displacement of the block is $x\sin\phi$.
Change in buoyancy: $\Delta B = \rho_l b^2 x \sin\phi g$

Using Newton's second law,
$ma = -\Delta B \sin\phi = -\left(\rho_l b^2 g \sin^2\phi\right)x$
Hence the period of oscillations is $ma = -\Delta B \sin\phi = -\left(\rho_l b^2 g \sin^2\phi\right)x$.
Angular frequency: $\omega^2 = \dfrac{\rho_l b^2 g \sin^2\phi}{m} = \dfrac{g \sin^2\phi}{b}$.
Period: $T = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{b}{g\sin^2\phi}} = 2\pi\sqrt{\dfrac{4b}{g}}$. (answer)

**《END 完》**