Pan Pearl River Delta Physics Olympiad 2012
2012 年泛珠三角及中华名校物理奥林匹克邀请赛
Part-1 (Total 6 Problems) 卷-1（共 6 题）
(9:00 am – 12:00 pm, 02-02-2012)

Q1 (5 points)

$E=\gamma mc^2$, where $\gamma \equiv \sqrt{\frac{1}{(1-\beta)(1+\beta)}} \approx \sqrt{\frac{1}{2(1-\beta)}}$.

So $mc^2 = E\sqrt{2(1-\beta)} < 10\cdot\sqrt{2\cdot 2\cdot 10^{-9}} = 632\,\text{eV}$. (4 points)

Upper limit 上限. (1 point)

Q2 (10 points)

设 X-Y 平面为水平面，单摆平衡时质点在（0，0），质点现在位置是（x，y）。利用小幅振动近似的结果，我们知道细绳的张力等于质点的重力，张力在 X-Y 平面的投影的大小为 $mg\frac{\sqrt{x^2+y^2}}{L}$，方向指向（0，0）。因此，X 方向的分力为

$F_x = -mg\frac{\sqrt{x^2+y^2}}{L}\frac{x}{\sqrt{x^2+y^2}} = -\frac{x}{L}mg$ (i)。

Y 方向的分力为 $F_y = -mg\frac{\sqrt{x^2+y^2}}{L}\frac{y}{\sqrt{x^2+y^2}} = -\frac{y}{L}mg$ (ii)。

动力学方程为 $-\frac{x}{L}g = \ddot{x}$，$-\frac{y}{L}g = \ddot{y}$。

通解为

$x(t) = A_x\cos(\omega t) + B_x\sin(\omega t)$， $y(t) = A_y\cos(\omega t) + B_y\sin(\omega t)$，

$\vec{r}(t) = x(t)\vec{x}_0 + y(t)\vec{y}_0$。 $\omega = \sqrt{g/L}$. (2 points)

设一般的初始条件 $\vec{r}(0) = X_0\vec{x}_0 + Y_0\vec{y}_0$， $\dot{\vec{r}}(0) = v_{x0}\vec{x}_0 + v_{y0}\vec{y}_0$， 则

$x(t) = X_0\cos(\omega t) + \frac{v_{x0}}{\omega}\sin(\omega t)$， $y(t) = Y_0\cos(\omega t) + \frac{v_{y0}}{\omega}\sin(\omega t)$. (2 points)

As we only ask for examples, there can be many different ways.题目只要求给出例子，所以可以有多种答案。

(a) $X_0 = D/2$, others are 0 其余为 0. Then 则 $x(t) = \frac{D}{2}\cos(\omega t)$， $y(t) = 0$. (2 points)

(b) $\vec{r}(0) = R\vec{x}_0$, $\dot{\vec{r}}(0) = R\omega\vec{y}_0$. Then $x(t) = R\cos(\omega t)$, $y(t) = R\sin(\omega t)$. (2 points)

(c) $\vec{r}(0) = a\vec{x}_0$, $\dot{\vec{r}}(0) = b\omega\vec{y}_0$. Then $x(t) = a\cos(\omega t)$, $y(t) = b\sin(\omega t)$. (2 points)

Q3 (9 points)

In the rotating reference frame, the force along the radial direction is 在跟着卫星转的旋转参照系里，沿半径方向的力为：

$F_r(r) = m\omega^2 r - \dfrac{GMm}{r^2}$, and $F_r(R) = m\omega_0^2 R - \dfrac{GMm}{R^2} = 0$.

After the impact, the total angular momentum is still conserved 碰撞后，角动量仍然守恒:

$0 = d(\omega r^2)_{r=R} = R^2 d\omega + 2\omega_0 R dr$.

Let the orbit radius change by *dr*, then 令轨道的变化为 *dr*

$dF_r(R) = m\omega_0^2 dr + 2\omega_0 R d\omega + m\dfrac{2GMm}{R^3} dr = -m\omega_0^2 dr$.

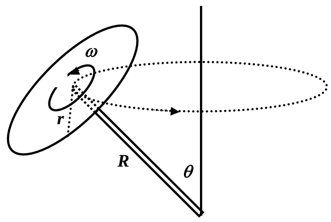
(Note that without taking into account the change of ω, the force is positive and the balance is unstable.若漏了考虑ω的变化，则力的变化是正的，原来的轨道运动变得不平衡了，那是不对的。)

This is a SHM with 上式结果显示卫星的径向运动是简谐振动，力常数为 $k = m\omega_0^2$, so 因此振动频率为 $\omega = \omega_0$.

The initial condition is 初始条件为 $v_0 = I/m$, and 和 $dr = 0$. So $dr(t) = \dfrac{-I}{m\omega_0}\sin(\omega_0 t)$.

$x(t) = [R - \dfrac{I}{m\omega_0}\sin(\omega_0 t)]\cos(\omega_0 t)$, $y(t) = [R - \dfrac{I}{m\omega_0}\sin(\omega_0 t)]\sin(\omega_0 t)$.

Q4 (6 points)



Angular momentum is 角动量为 $J = mr^2\omega/2$.

Torque 力矩 $\tau = mgR\sin\theta$, and pointing perpendicular to the paper plane 方向垂直于纸面.

The change of angular momentum is 角动量的变化为 $\Delta J = (J\sin\theta)\Delta\phi$.

$\tau = \dfrac{\Delta J}{\Delta t} = (J\sin\theta)\dfrac{\Delta\phi}{\Delta t} = (J\sin\theta)\Omega$. So $\Omega = \dfrac{2gR}{r^2\omega}$.

Q5 (10 points)

Let the image charge be at *x* on the X-axis 放个镜像电荷在 X-轴上 *x* 点处。

$$\Phi(R) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\sqrt{R^2 + x^2 + 2Rx\cos\theta}} - \frac{q_1}{\sqrt{R^2 + b^2 + 2Rb\cos\theta}}\right) = 0$$

$\Rightarrow q^2(R^2 + b^2 + 2Rb\cos\theta) = q_1^2(R^2 + x^2 + 2Rx\cos\theta)$。

上式对任何角度θ都成立，所以有

$q^2(R^2 + b^2) = q_1^2(R^2 + x^2)$ (i)，

$q^2 b = q_1^2 x$ (ii)。

将 *b* 代掉，得

$x^2 q_1^4 - q^2(R^2 + x^2)q_1^2 + q^4 R^2 = 0$ (iii)。

解上式，得两个解。第一个是 $q_1 = -q$， $b = x$，即把球外的点电荷中和掉。因为这个镜像电荷没有放在球面内，不在所考虑的解的空间以外，所以不能用，舍去。第二个是 $q_1 = -Rq/x$， $b = R^2/x$。这

就是镜像电荷的值和位置。(It is OK if answers are given without derivations.若没有上述推导而只有答案也得全分。)

In order to make the potential on the plane zero, we need two more image charges, namely $q_3 = -q_1$ at $-R^2/x$, and $q_3 = -q$ at $-3R/2$. One can see that q_2 and q_3 combined will make the potential on the sphere surface zero. One can see that q_2 and q_3 combined will make the potential on the sphere surface zero. 为了使平面的电势为 0，我们需要另外两的镜像电荷，$q_3 = -q_1$ 在 $-R^2/x$, and $q_3 = -q$ 在 $-3R/2$. 而 q_2 、q_3 合起来也使球面的电势为 0。(2 points)

电荷受的力为：$F_1 = \frac{qq_1}{4\pi\varepsilon_0}\frac{1}{\left(x-b\right)^2} = \frac{-q^2}{4\pi\varepsilon_0}\frac{R}{x\left(x-R^2/x\right)^2} = \frac{-q^2}{4\pi\varepsilon_0}\frac{xR}{\left(x^2-R^2\right)^2}$。

现在求 q 的电势，也就是把 q 从无穷远拉到现在位置所需的能量。用作用力做功的方法，

$$W_1 = \int_{3R/2}^{\infty} F_1 dx = \frac{-q^2 R}{4\pi\varepsilon_0}\int_{3R/2}^{\infty}\frac{x}{\left(x^2-R^2\right)^2}dx = \frac{-q^2}{8\pi\varepsilon_0}\left(\frac{R}{d^2-R^2}\right) = \frac{-q^2}{8\pi\varepsilon_0 R}\left(\frac{4}{9-4}\right) = \frac{-q^2}{10\pi\varepsilon_0 R}$$ 。(4 points)

读者请注意，此势能和直接用电势所得的值是不同的。q 所在位置的电势由 q_1 产生，其值为

$U = \frac{q_1}{4\pi\varepsilon_0}\frac{1}{\left(d-b\right)} = \frac{-q}{4\pi\varepsilon_0}\frac{R}{\left(d^2-R^2\right)} = \frac{-q}{5\pi\varepsilon_0 R}$. 所以 q 的势能为 $W = \frac{-q^2}{5\pi\varepsilon_0 R}$.

正确答案是错误答案的一半。两者差别的主要原因，是因为镜像电荷的值随真电荷的位置而变。所以计算电荷势能最可靠的方法是用作用力的路径积分来做。

$$W_2 = \int_{3R/2}^{\infty} F_2 dx = \frac{q^2 R}{4\pi\varepsilon_0}\int_{3R/2}^{\infty}\frac{x}{\left(x^2+R^2\right)^2}dx = \frac{q^2}{8\pi\varepsilon_0 R}\left(\frac{4}{9+4}\right) = \frac{q^2}{26\pi\varepsilon_0 R}.$$

$$W_3 = \int_{3R/2}^{\infty} F_3 dx = \int_{3R/2}^{\infty}\frac{q^2}{16\pi\varepsilon_0 x^2}dx = -\frac{q^2}{24\pi\varepsilon_0 R}$$ 。(2 points)

$$W = W_1 + W_2 + W_3 = \frac{q^2}{2\pi\varepsilon_0 R}\left(-\frac{1}{5}+\frac{1}{13}-\frac{1}{12}\right) = \frac{-161}{1560}\frac{q^2}{\pi\varepsilon_0 R}.$$ (1 point)

Q6 (10 points)

(a) Let $c = \frac{3}{2}$, so that $c_v = cnR$. The expansion is an adiabatic process, 膨胀过程为绝热过程，所以有 $PV^{1+1/c} = con.$

The work done to the piston is 对活塞做的功为

$$W = \int_{V_0}^{\kappa V_0} PdV = P_0 V_0^{1+1/c}\int_{V_0}^{\kappa V_0} V^{-1-1/c}dV = cP_0V_0(1-\kappa^{-1/c}) = \frac{3}{2}nRT_0(1-\kappa^{-2/3}).$$ (2 points)

Maximum work is 最大功为 $W_{max} = \frac{3}{2} nRT_0$, which is the total internal energy of the gas 等于气体的总内能，也就是气体可做的最大功. (1 point)

(b) $\frac{\Delta S}{nR} = \ln(\kappa) + c\ln(T/T_0) = \ln(\kappa) + c\ln\left(\frac{1}{T_0}(T_0 - \frac{\eta W}{cnR})\right)$

Put in the answer in (a) we get 代入(a) 里功的表达式，得 $\frac{\Delta S}{nR} = \ln(\kappa) + \frac{3}{2}\ln\left(1-\eta+\eta\kappa^{-2/3}\right)$. (5 points)

To see if ΔS is positive, let 令 $\Delta S = cnR\ln(m)$, so $m-1 = (1-\eta)(\kappa^{-2/3}-1) \geq 0$, and $\Delta S > 0$. (1 point)

(c) When $\eta = 1$, $\Delta S = 0$, which is consistent with the fact that there is no net loss in internal energy of the gas. 当 $\eta = 1$, $\Delta S = 0$, which is consistent with the fact that there is no net loss in internal energy of the gas 符合气体自由膨胀时熵不变这一结果。(1 point)

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2012 年泛珠三角及中华名校物理奥林匹克邀请赛
Part-2 (Total 3 Problems) 卷-2（共 3 题）
(2:30 pm – 5:30 pm, 02-02-2011)

Q1 Stella Interferometer (8 points)

(a) The difference between the path differences of the two pairs of split waves is $\Delta = D\cdot\delta\theta$. When $\Delta = \lambda/2$, the bright fringes of one star coincide with the dark fringes of the other. 两个平面波在两个入口镜的光程差的差为 $\Delta = D\cdot\delta\theta$. 当 $\Delta = \lambda/2$ 时, 一颗星的亮条纹与另一颗星的暗条纹刚好重叠，所以： $D = \dfrac{\lambda}{2\delta\theta} = \dfrac{180\cdot 5.0\cdot 10^{-7}}{2\cdot 3.14\cdot 3\cdot 10^{-6}}m = 4.8\ m$. (6 points)

(b) $\delta\theta = \dfrac{1.22\lambda}{D}$, so $D = \dfrac{1.22\lambda}{\delta\theta} = \dfrac{1.22\cdot 180\cdot 5.0\cdot 10^{-7}}{3.14\cdot 3\cdot 10^{-6}}m = 12\ m$. (2 points)

Q2 Y-particle (12 points)

(a) The two B-mesons should have the same momentum and energy because the original Y-meson is at rest. The kinetic energy of one B-meson is
$E_k = \frac{1}{2}(10.58 - 2\times 5.28) = 0.01$ GeV, which is much less than the rest energy of B-mesons.
So we can use $E_k = \frac{1}{2}m_B v_0^2$. Putting the numbers in we get $v_0 = \sqrt{\frac{0.01}{5.28}}\ c = 0.0615c$. As the B-mesons are moving at low speed their lifetime change can be ignored. So
$L_0 = v_0\tau_0 = 3\times 10^8 \times 0.0615\times 1.5\times 10^{-12} = 0.028$ mm.
由于 Y 介子是静止的，所以两个 B 介子的动量相等，方向相反。B 介子的动能为
$E_k = \frac{1}{2}(10.58 - 2\times 5.28) = 0.01$ GeV, 比它的静止质量小亨多，所以可用经典力学
$E_k = \frac{1}{2}m_B v_0^2$ 来求它的速度。将数值代入后得 $v_0 = \sqrt{\frac{0.01}{5.28}}\ c = 0.0615c$. B 介子寿命因运动而延长的效应可忽略，因此
$L_0 = v_0\tau_0 = 3\times 10^8 \times 0.0615\times 1.5\times 10^{-12} = 0.028$ mm. (2 points)

(b) Rough estimation: the speed of the B-mesons should be $v_0 L/L_0 = 0.44c$, so we need precise formula which takes into account the lifetime change. 粗略估计: B 介子的速度为 $v_0 L/L_0 = 0.44c$, 所以必须考虑相对论效应.

$$L = \frac{v\tau_0}{\sqrt{1-(v/c)^2}}.\ \text{So}\ v = \frac{c}{\sqrt{(c\tau_0/L)^2+1}} = 0.406c.$$

$$P = \gamma m_B v = 0.406\times 1.094\times 5.28(GeV/c) = 2.35\,\text{GeV/c}\ \text{(3 points)}$$

(c) The Y-mesons move with the center-of-mass (CoM) frame of the B-mesons. From (a) we get the speed of the B-mesons in that frame as $v_0 = 0.0615c$. From (b) we know that in the laboratory frame the speed of the B-mesons is $v = 0.406c$. Let the relative speed between the CoM frame and the laboratory frame be –*u*, which is also the speed of the Y-mesons in the laboratory frame, then $v=\frac{v_0+u}{1+v_0u/c^2}$. Solving it we get *u* = 0.336*c*. Y 介子的速度与 B 介子质心的速度一致，由 (a) 我们知道 B 介子在质心参照系的速度为 v_0 = 0.0615*c*. 由(b) 我们知道 B 介子在实验室参照系里的速度为 *v* = 0.406*c*. 令质心参照系相对于实验室的速度（也是 Y 介子在实验室的速度）为–*u*, 则 $v=\frac{v_0+u}{1+v_0u/c^2}$. 由此得 *u* = 0.336*c*. The total energy of the Y-mesons in the laboratory frame is Y 介子的总能量为 $E_Y=m_Yc^2/\sqrt{1-(u/c)^2}=1.062m_Yc^2$. (4 points)

(d) In the CoM an electrons has equal and opposite momentum as a positron, while together they must have the energy to create a Y-meson. The momentum 4-vector is then $\begin{pmatrix}0\\ m_Yc\end{pmatrix}$ in CoM. 在 Y 介子参照系里电子和正电子的动量相等，方向相反，总能量等于 Y 介子的静质量。因此 Y 介子的动量-能量 4 矢为 $\begin{pmatrix}0\\ m_Yc\end{pmatrix}$。In the laboratory frame the momentum 4-vector is 在实验室参照系，该 4 矢为

$$\begin{pmatrix}\gamma & \beta\gamma\\ -\beta\gamma & \gamma\end{pmatrix}\begin{pmatrix}0\\ m_Yc\end{pmatrix}=\begin{pmatrix}\beta\gamma m_Yc\\ \gamma m_Yc\end{pmatrix}.$$

Let the momenta of electron and positron be P_1 and P_2, respectively, and also note that as the electron (and positron) rest energy (0.511 MeV) is thousands of times less than that of the Y-meson, so the energy of an electron is simply cP_1. 令电子和正电子的动量分别为 P_1 、P_2，又由于现在电子的能量远大于它的静止质量 (0.511 MeV)，所以它的能量为 cP_1。We then have 由此我们得

$$\begin{pmatrix}P_1-P_2\\ P_1+P_2\end{pmatrix}=\begin{pmatrix}\beta\gamma m_Yc\\ \gamma m_Yc\end{pmatrix}$$, where 其中 $\beta\equiv u/c$.

The electron energy in the laboratory frame is then 电子在实验室的能量为

$$cP_1=\frac{1}{2}(1+\beta)\gamma m_Yc^2=\frac{1}{2}\times1.336\times1.062\times m_Yc^2=0.709m_Yc^2=7.51\ \text{GeV}.$$

正电子的能量为

$$cP_2=\frac{1}{2}(1-\beta)\gamma m_Yc^2=\frac{1}{2}\times0.664\times1.062\times m_Yc^2=0.353m_Yc^2=3.73\ \text{GeV}$$, or vise versa.

(3 points)

Q3 Penning Trap (30 points)

(a) $qBv=m\frac{v^2}{r_0}$, so $\omega_c=\frac{v}{r_0}=\frac{qB}{m}$. $E_k=\frac{1}{2}mv^2=\frac{1}{2}m\omega_c^2r_0^2$. (1 point)

(b) $x=\frac{D}{2}-r\cos(\omega_ct-\phi)$, $Q_1-Q_2=-2q\frac{r}{D}\cos(\omega_ct-\phi)$.

$I=\frac{dQ}{dt}=2q\frac{r_0\omega_c}{D}\sin(\omega_ct-\phi)$. Larger r_0 leads to larger current. (2 points)

(c) The energy gained in each cycle is 每周期得到的能量为

$$dE_k = \int_{T_c} qE_0 \cos(\omega_c t) \cdot r\omega_c \cos(\omega_c t)dt = \frac{1}{2} qE_0 r\omega_c T_c \text{, and } \frac{dE_k}{dt} = \frac{1}{2} qE_0 r\omega_c \text{ (1 point)}$$

On the other hand, from (a) we have 由(a) 我们得 $\frac{dE_k}{dt} = m\omega_c^2 r \frac{dr}{dt}$. (1 point)

So $m\omega_c^2 r \frac{dr}{dt} = \frac{1}{2} qE_0 r\omega_c$, and $R = r_0 + \frac{qE_0}{2m\omega_c} T$ (1 point)

(d) $E_z = -2V_0 \frac{z}{z_0^2}$. $\omega_z^2 = \frac{2qV_0}{mz_0^2}$. (2 points)

(e) $\nabla^2 V(\vec{r}) = 0$ so $\beta = -1/2$. (1 point)

Use ω_c and ω_z as known for the remaining part of the question.

(f) The electric field in the x-y plane is 沿 X-Y 平面的电场为 $\vec{E} = -V_0 \frac{x\vec{x}_0 + y\vec{y}_0}{z_0^2}$. (1 point)

The equation is 粒子的运动方程为

$$m(\ddot{x}\vec{x}_0 + \ddot{y}\vec{y}_0) = eV_0 \frac{x\vec{x}_0 + y\vec{y}_0}{z_0^2} + eB(\dot{x}\vec{x}_0 + \dot{y}\vec{y}_0) \times \vec{z}_0 = eV_0 \frac{x\vec{x}_0 + y\vec{y}_0}{z_0^2} - eB(\dot{x}\vec{y}_0 - \dot{y}\vec{x}_0)$$

(1 point)

$$\ddot{x} - \omega_c \dot{y} - \frac{1}{2}\omega_z^2 x = 0$$

$$\ddot{y} + \omega_c \dot{x} - \frac{1}{2}\omega_z^2 y = 0 \qquad \text{(2 points)}$$

(g) Multiply i to the first equation, and add to the second one, 将第一式乘 i 后与第二式相加，得 we get $\ddot{u} + ia\dot{u} + bu = 0$, where $a = \omega_c$ and $b = -\frac{1}{2}\omega_z^2$. (2 points)

(h) $\omega_{\pm} = \frac{\omega_c \pm \sqrt{\omega_c^2 - \omega_z^2}}{2}$ (2 points)

(i) $x(0) = R$, $y(0) = 0$, $\dot{x}(0) = 0$, $\dot{y}(0) = -R\omega_c$ (2 points)

Then $R = A_+ + A_-$, $R\omega_c = \omega_+ A_+ + \omega_- A_-$ (1 point)

Solving the equations we get 解上述方程，得 $A_- = \frac{\omega_+ - \omega_c}{\omega_+ - \omega_-} R = -\frac{1}{2}\left(\frac{\omega_z}{\omega_c}\right)^2 R$,

$A_+ = \frac{\omega_c - \omega_-}{\omega_+ - \omega_-} R = R$ (2 points)

(j) $\tilde{x} = x\cos(\Omega t) + y\sin(\Omega t)$, $\tilde{y} = y\cos(\Omega t) - x\sin(\Omega t)$. So $\tilde{u} = ue^{-i\Omega t}$. (3 points)

(k) $\tilde{u} = ue^{i\omega_- t} = A_- + A_+ e^{-i(\omega_+ - \omega_-)t}$, which is a circle centered at $x = A_-$.这是一个中心在 $x = A_-$ 的圆。(1 point)

(l) (k)中的圆心绕原点作圆周运动。 (1 point)

(m) $\tilde{u} = A_+ e^{-i\omega_1 t} + A_- e^{i\omega_1 t}$ where $\omega_1 = \sqrt{\omega_c^2 - \omega_z^2}$. (1 point)

So $\tilde{x} = (A_+ + A_-)\cos(\omega_1 t)$, $\tilde{y} = (A_+ - A_-)\sin(\omega_1 t)$, which is an ellipse with $(A_+ + A_-)$ being one axis and $(A_+ - A_-)$ being the other. Under more special conditions, the ellipse can become a line or a circle. 这是个椭圆，在特定条件下可变成圆(A_- 或 A_+ 等于 0)，或直线($A_- = \pm A_+$)（(2 points)

《THE END 完》