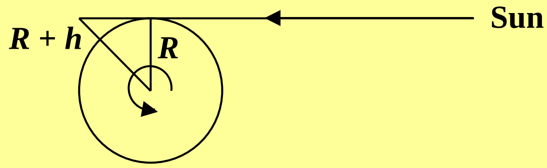
Part-1 (Total 6 Problems) 卷-1（共6 题）

1. Sunset Twice a Day (6 points) 一天两观日落 （6 分）

Presently the tallest tower in the world is Burj Khalifa in Dubai. Its height is 828 m. An Internet news article reported that one can watch sunset twice in one day with this tower.
杜拜的哈利法塔是现时世界上最高的建筑，高度为 828 米。互联网上有新闻文章报导，可以利用这塔在一天内两次观看到日落。

(a) What is the time range of sunset between the bottom and the top of the tower? Give your answer in minutes. Parameters: Earth's radius = 6400 km. Distance between Sun and Earth = 1.5×10^{11} m. (3 points)
试求塔底和塔顶之间日落时间的范围。答案请以分钟为单位。参数：地球半径 = 6400 公里。太阳和地球之间的距离 = 1.5×10^{11} 米。（3 分）



Neglecting the tilt of Earth's axis and the latitude of Dubai, the angular displacement of Earth between the two sunsets at the bottom and top of the tower
假设地球转轴的倾角和杜拜的纬度可略，则在塔底和塔顶两次日落之间地球的角位移为

$$\sin\theta = \frac{\sqrt{(R+h)^2 - R^2}}{R+h} \qquad [1]$$

$$\approx \frac{\sqrt{2Rh}}{R} = \sqrt{\frac{2h}{R}} << 1 \quad \Rightarrow \quad \theta \approx \sqrt{\frac{2h}{R}} = \sqrt{\frac{(2)(828)}{6400\times10^3}} = 0.0161\,\text{radian} \qquad [1]$$

Time between the two sunsets 两次日落之间的时间

$$= \left(\frac{0.0161}{2\pi}\right)(24)(60)\ \text{min} = 3.7\ \text{min} \qquad [1]$$

(b) Burj Khalifa also has the world's third fastest elevator (lift) with a speed of v = 10 m/s. Immediately before the elevator starts moving upwards at the speed v from the bottom of the tower, a tourist in the elevator views the sunset. When he reaches the observatory at the height of 452 m, he found that the Sun has risen. Calculate the inclination angle of the Sun above the horizon. Give your answer in degrees. (3 points)
哈里发塔还拥有世界第三快的电梯，速度可达 v = 10 m/s。有电梯内的游客，在电梯从塔底开始上升前一瞬看到日落，其后电梯以速度 v 上升。当他到达在 452 米高度的观景台时，发现太阳上升了。试计算太阳在地平线以上的仰角。答案请以度为单位。（3 分）

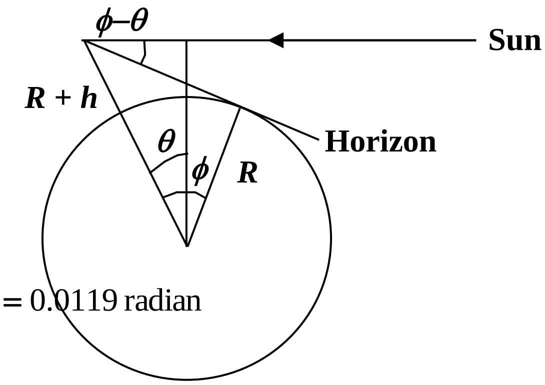
Time to travel to the observatory 前往观景台的时间 $t = \dfrac{h}{v}$

Earth's angular speed 地球的角速度 $\omega = \dfrac{2\pi}{(24)(60)(60)}$

Earth's angular displacement 地球的角位移

$$\theta = \omega t = \frac{2\pi}{(24)(60)(60)}\left(\frac{452}{10}\right) = 0.00329\ \text{radian}$$

[1]



Change in the horizon 地平线的改变

$$\cos\phi = \frac{R}{R+h} \Rightarrow$$

$$\phi \approx \sin\phi = \sqrt{1-\left(\frac{R}{R+h}\right)^2} \approx \sqrt{\frac{2h}{R}} = \sqrt{\frac{(2)(452)}{6400\times10^3}} = 0.0119\ \text{radian}$$

[1]

Elevation angle 仰角

$$\phi-\theta \approx 0.0119 - 0.0033 = 0.0086\ \text{radian} = 0.49^{\circ}$$ [1]

Remark: This is roughly the angular size of the Sun. So the tourist can view almost the entire Sun.

备注：这大致是太阳的角直径。因此，游客可以看到几乎整个太阳。

2. Radiocarbon Dating (5 points) 放射性碳年龄测定法 （5 分）

Radiocarbon dating is a technique used in archeology to estimate the age of organic materials, such as wood and leather. It uses the fact that the density of ^{14}C atoms in the atmosphere is constantly around 1.3 atoms of ^{14}C in every 10^{12} atoms of all isotopes of carbon. However, when an organism dies, ^{14}C cannot be replenished and decreases due to β decay with a half-life of 5730 years. The radioactive decay can be written in the following form:

放射性碳年龄测定法是考古学上用来估计有机物料（如木材和皮革）年龄的技术。它的根据，在于 ^{14}C 原子在大气中，浓度恒常处于每 10^{12} 粒碳原子中（包括所有同位素）有 1.3 粒 ^{14}C 原子。但是，生物死亡后，^{14}C 不能得到补充，并因β衰变逐渐降低，半衰期为 5730 年。这放射性衰变可以写成以下形式：

$$^{14}_{6}C \rightarrow {}^{14}_{7}N + e^- + \bar{\nu}_e$$

(a) Suppose we obtain 50 grams of carbon from a piece of wood dated back to a prehistoric tomb. Using the carbon average atomic mass of 2×10^{-26} kg, calculate the number N_0 of ^{14}C atoms when the wood was still part of a living tree. (1 point)

假设我们从史前古墓的一块木头得到 50 克碳。已知碳的平均原子质量为 2×10^{-26} 千克，试计算木材仍是活树一部分时，^{14}C 原子的数目 N_0。（1 分）

$$N_0 = \left(\frac{50\times10^{-3}}{2\times10^{-26}}\right)(1.3\times10^{-12}) = 3.25\times10^{12}$$ [1]

(b) We can determine the age of the tomb if we know the number N of ^{14}C atoms from the 50 grams of carbon. There is no way to directly count the number of ^{14}C atoms, but we detect a total of 935 electrons emitted from the 50 grams of carbon in 10 minutes. How old is the tomb? (3 points)

要估算古墓的年代，我们需要知道该 50 克碳中 ^{14}C 原子的数目 N。我们无法直接数算 ^{14}C 原子的数目，但我们发现 50 克碳在 10 分钟内放射了共 935 粒电子。古墓的年龄是多少？（3 分）

$N(t) = N_0 e^{-\frac{t}{\tau}}$

$\frac{1}{2} = e^{-t_{1/2}/\tau} \Rightarrow \tau = \frac{t_{1/2}}{\ln 2} = 8267$ years [1]

Decay rate 衰变率 $R(t) = -\frac{dN}{dt} = \frac{N_0}{\tau} e^{-\frac{t}{\tau}} \Rightarrow t = -\tau \ln\left(\frac{R\tau}{N_0}\right)$ [1]

$t = -8267 \ln\left(\frac{(93.5)(8267 \times 365 \times 24 \times 60)}{3.25 \times 10^{12}}\right) = 17190$ years [1]

(c) An archaeologist claims that he/she discovered a fossil plant with an age of 2×10^8 years using the method of radiocarbon dating. A scientist says that this result is nonsense. Which side will you stand on? Please explain your reasons. (1 point)
某考古学家声称，他/她利用放射性碳年龄测定法，发现年代为 2×10^8 年的化石植物。某科学家说，这结果是无稽之谈。你认为哪方较合理？请解释你的理由。（1 分）

The scientist is more reasonable. To see this, let us do a calculation basing on 50 grams of carbon:
科学家较合理。要了解这一点，让我们根据 50 克碳作一计算：

$N(t) = N_0 e^{-\frac{t}{\tau}} = (3.25 \times 10^{12}) \exp\left(-\frac{2 \times 10^8}{8267}\right) = 2.27 \times 10^{-10495} << 1.$

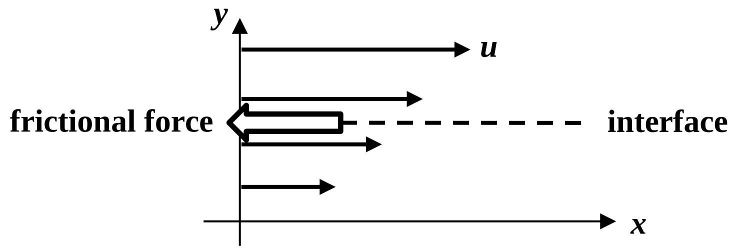
This is impossible to be detected. ($N \geq \sqrt{N} \Rightarrow N \geq 1$ for shot noise limited perfect detection.)
In order to have at least 1 ^{14}C atom left today, the archaeologist needs at least $50\,\text{g} \times 2 \times 10^{10483} = 10^{10482}\,\text{kg}$, which is impossible to be obtained (it is more than the mass of the Earth).
这是不可能被检测出来的。 （$N \geq \sqrt{N} \Rightarrow N \geq 1$，是散粒噪声对準确检测的限制。）
若要至少有 1 粒 ^{14}C 原子到今天仍然存留，考古学家至少需要 $50\,\text{g} \times 2 \times 10^{10483} = 10^{10482}\,\text{kg}$（这质量比地球质量更大）。 [1]

3. Viscosity (7 points) 粘度 （7 分）

When uneven forces are applied to a fluid, the flow velocities at different locations will be different. For viscous fluids, frictional forces will be present when two adjacent layers of fluids flow at different velocities. As shown in the figure, the viscosity μ of the fluid is defined by the equation:
当不均匀的力施加到流体中，流速在不同的位置将是不同的。对于粘性流体，当相邻的两层流体以不同速度流动时，摩擦力便会存在。如下图所示，流体的粘度μ由下式定义:

$$F = -\mu \frac{du}{dy} \Delta A,$$

where F is the frictional force experienced by the fluid at an interface of area ΔA in the x direction, u is the x component of the velocity and du/dy is the velocity gradient. In this problem, we will analyze the viscosity using the kinetic theory of gases.
其中F是流体在x方向、面积为ΔA的界面上的摩擦力，u是方向的速度，du/dy是速度梯度。在这问题中，我们将以气体运动理论分析流体的粘度。



Let τ be the average time between successive collisions of a gas molecule with other molecules. Molecule i moves with velocity $\vec{v}_i$ in random directions, and the average velocity at height y is $\vec{u}(y)$.

设τ为气体分子与其他分子连续碰撞之间的平均时间。分子i以速度$\vec{v}_i$沿随机方向运动，而在高度y的平均速度为$\vec{u}(y)$。

(a) Suppose the interface is at a height y. What is the average x component of the momentum at height $y + \Delta y$? (1 point)

假设界面高度为y。在高度$y + \Delta y$的动量，其平均x分量是多少？（1分）Average x-momentum at height $y + \Delta y$ 在高度$y + \Delta y$的动量，其平均x分量是

$$= m\left(u + \frac{du}{dy}\Delta y\right). \qquad [1]$$

(b) An incident molecule arrives at height y. The y component of its velocity is v_y. What is Δy of the height where the molecule experiences the collision last time? (1 point)

一分子入射到高度y。其速度的y分量为v_y。分子上一次遇到碰撞的高度的Δy是什么？（1分）

Height where the incident molecule experiences the collision last time $\Delta y = -v_y\tau$.

分子上一次遇到碰撞的高度 $\Delta y = -v_y\tau$。 [1]

(c) Compared with the average x component of the momentum of the gas molecules at the interface, what is the average extra x-momentum carried by the incident molecules of a given v_y when it arrives at height y? (1 point)

当给定v_y的入射分子到达高度y时，它的平均额外x-动量是什么（与界面上的气体分子动量的平均 x 分量相比）？ （1分）

Average extra x-momentum平均额外x-动量 $m\left(u - \frac{du}{dy}v_y\tau\right) - mu = -\tau m v_y \frac{du}{dy}$ [1]

(d) The gas contains n molecules per unit volume. What is the rate of x-momentum transfer through an area ΔA? Hence find an approximate expression for the viscosity of the fluid according to the kinetic theory of gases. How does the viscosity depend on temperature? (4 points)

气体单位体积含有n粒分子。通过面积ΔA的x-动量，传递率是什么？试根据气体运动理论，由此推导流体粘度的近似表达式。粘度与温度有何关系？（4分）

Number of incident molecules passing through the area per unit time $= n v_y \Delta A$.

每单位时间入射分子通过面积的数目 $= n v_y \Delta A$

Each molecule transport an x-momentum equal to $-\tau m v_y \dfrac{du}{dy}$.

每个分子运输的x-动量等于 $-\tau m v_y \dfrac{du}{dy}$。

Hence the rate of x-momentum transfer in the upward direction is
因此，x-动量向上的传递率是

$$-\left(\tau m v_y \frac{du}{dy}\right)(n v_y \Delta A) = -\tau n m v_y^2 \frac{du}{dy} \Delta A \qquad [1]$$

Using Newton's second law of motion, frictional force experienced by the layer above the interface is
利用牛顿第二运动定律，界面上层受到的摩擦力为

$$F = -\tau n m \langle v_y^2 \rangle \frac{du}{dy} \Delta A \quad \Rightarrow \quad \mu = \tau n m \langle v_y^2 \rangle = \frac{1}{3} \tau n m \langle v^2 \rangle \qquad [1]$$

According to the kinetic theory of gases, 根据气体运动理论，

$$\frac{1}{2} m \langle v^2 \rangle = \frac{3}{2} kT \quad \Rightarrow \quad \mu = \tau n k T \qquad [1]$$

Viscosity is proportional to T. 粘度与T成正比。 [1]

Remark 1: The expression derived by using the Maxwell-Boltzmann distribution is different only in the coefficient.
备注1：通过使用麦克斯韦 - 玻尔兹曼分布得出的表达式，仅在系数有分别。

Remark 2: In practice, τ is also temperature dependent since $\tau = \lambda/v$, where λ is the mean free path that is mainly dependent on the density of gas, and v is proportional to $\sqrt{T}$. Combining, μ is proportional to $\sqrt{T}$.
备注2：在现实中，τ也随溫度改变，因为$\tau = \lambda/v$，其中λ是平均自由程，主要依赖於氣體的密度，v与$\sqrt{T}$ 成正比。結合后，μ与$\sqrt{T}$ 成正比。

4. Age of the Universe (10 points) 宇宙的年龄 （10 分）

Hubble discovered that the velocities v of galaxies receding from Earth are proportional to their distance d from Earth,
哈勃发现星系远离地球的速度 v 与地球距离 d 成正比，

$$v = H_0 d ,$$

where H_0 is the Hubble constant at the present age of the universe. It was recently measured to be 68 km/s/Mpc.
其中 H_0 为宇宙目前的哈勃常数。最近测得为 68 km/s/Mpc。

(a) Assuming that the universe expanded from the beginning to the present at a uniform speed, estimate the age of the universe. Give your answer in billion years. Parameters: 1 Mpc = 3.26×10^6 light years, speed of light = 300,000 km/s. (2 points)
假设宇宙从太初到现在以均匀速率膨胀，试估计宇宙的年龄。答案请以 billion years（十亿年）为单位。参数：1 Mpc = 3.26×10^6 光年，光速 = 300,000 km/s。 （2 分）

At uniform rate, 在均匀速率下，$v = \frac{d}{t}$.

Substituting into Hubble's law, 代入哈勃定律，$\frac{d}{t} = H_0 d \quad \Rightarrow \quad t = \frac{1}{H_0}$ [1]

$= \frac{3.26\times10^6\times300{,}000}{68} = 14.4\ \text{billion years}$ [1]

(b) However, the universe does not expand at a speed uniform in time due to the gravitational attraction of matter. Friedmann modeled the universe as an expanding sphere of matter with uniform density $\rho(t)$ at time t. Consider a test mass m on the surface of the sphere of radius $r(t)$ at time t. The total energy of the test mass is mU. Find the relation between the expansion velocity $v(t)$ and radius $r(t)$ at time t based on Newtonian mechanics. You may use G to represent the universal gravitational constant. (1 point)

但是，由于物质的万有引力，宇宙膨胀的速率在时间上不是均匀的。弗里德曼模拟宇宙为一膨胀中的均匀密度球体，在时间 t 其密度是$\rho(t)$。考虑在时间 t 时，在半径为 $r(t)$的球体表面上有一测试质量 m。测试质量的总能量为 mU。根据牛顿力学，找出在时间 t 的膨胀速度 $v(t)$和半径 $r(t)$之间的关系。你可用 G 代表万有引力常数。（1 分）

Since the attraction due to matter outside the sphere vanishes, the gravitational potential energy of the test mass is only due to matter inside the sphere. Using the conservation of energy,

在球体外的引力抵消，测试质量的引力势能只需考虑球体内的物质。利用能量守恒定律，

$\frac{1}{2}mv^2 - \frac{Gm}{r}\left(\frac{4}{3}\pi r^3\rho\right) = mU \quad \Rightarrow \quad \frac{1}{2}v^2 - \frac{G}{r}\left(\frac{4}{3}\pi r^3\rho\right) = U$ [1]

(c) Recent satellite data shows that U is negligible. In this case, the expansion of the universe is described by the power-law $\frac{r(t)}{r_0} = \left(\frac{t}{t_0}\right)^n$, where r_0 and t_0 are the present values of $r(t)$ and t respectively. Find n and t_0. Express your answer in terms of G and the density ρ_0 of the present universe. (4 points)

最近的卫星数据显示，U 可以忽略不计。在这情况下，宇宙的膨胀可用幂律 $\frac{r(t)}{r_0} = \left(\frac{t}{t_0}\right)^n$ 描述，其中 r_0 和 t_0 分别为 $r(t)$ 和 t 的现值。求 n 和 t_0。答案请以 G 和宇宙密度的现值ρ_0 表达。（4 分）

When $U = 0$, 当 $U = 0$, $\frac{1}{2}v^2 = \frac{G}{r}\left(\frac{4}{3}\pi r^3\rho\right)$.

Note that 注意 $\rho(t) = \rho_0\left(\frac{r_0^3}{r(t)^3}\right)$.

Hence 所以 $v^2 = \frac{8\pi G r_0^3\rho_0}{3r}$. [1]

$v = \frac{dr}{dt} = \frac{nr_0}{t_0}\left(\frac{t}{t_0}\right)^{n-1}$. [1]

Substituting, 代入上式，$\frac{n^2 r_0^2}{t_0^2}\left(\frac{t}{t_0}\right)^{2n-2} = \frac{8\pi G r_0^2 \rho_0}{3}\left(\frac{t}{t_0}\right)^{-n}$.

Comparing exponents and coefficients, 比较指数和系数，

$n = \frac{2}{3}$ [1]

$t_0 = \frac{1}{\sqrt{6\pi G\rho_0}}$ [1]

(d) Express the present age of the universe in terms of the present value of the Hubble constant. Estimate the age of the universe in this Newtonian picture. Give your answer in billion years. Based on your understanding about current developments in physics research, how is this result different from the current estimate of the age of the universe? (3 points)
试以哈勃常数的现值，表达宇宙目前的年龄。试以此牛顿力学的角度，估计宇宙的年龄。答案请以 billion years（十亿年）为单位。根据你对物理学研究当代发展的理解，这结果与当前对宇宙年龄的估计有何不同？（3 分）

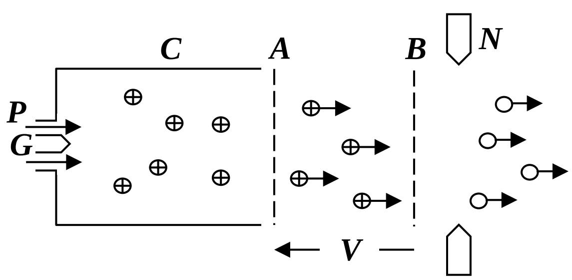
Since $\frac{r(t)}{r_0} = \left(\frac{t}{t_0}\right)^{\frac{2}{3}}$, $v = \frac{dr}{dt} = \frac{2r_0}{3t_0}\left(\frac{t}{t_0}\right)^{-\frac{1}{3}}$. At the present age, $t = t_0$. Hence $v_0 = \frac{2r_0}{3t_0}$. [1]

$H_0 = \frac{v_0}{r_0} = \frac{2}{3t_0} \Rightarrow t_0 = \frac{2}{3H_0} = \frac{(2)(3.26\times10^6\times300{,}000)}{(3)(68)} = 9.6\text{ billion years}$ [1]

This is less than the present estimate of the age of the universe (14 billion years). [1]

5. Electrostatic Ion Thrusters (12 points) 静电离子推进器 （12 分）

Electrostatic ion thrusters are used in spacecraft to control their trajectories in space. Its operating principle is shown in the following figure.
静电离子推进器用于控制航天器在太空的轨迹。它的工作原理如下图所示。



Streams of propellant atoms P are injected into the chamber C. The rate of injection is R, measured in the number of atoms per unit time. The atoms are ionized by bombarding with electrons shot from electron gun G. The positive ions are accelerated from grid electrode A to grid electrode B by the accelerating voltage V between them. The neutralizing electrode N emits electrons to neutralize the ion beam, preventing the spacecraft from gaining a net negative charge.
推进剂原子 P 被喷注入腔室 C。喷注的速率为 R，R 的单位为单位时间内的原子数目。原子被从电子枪 G 射出的电子碰撞而离子化。栅电极 A 到栅电极 B 之间的加速电压 V，使正离子加速。中和电极 N 发射电子，把离子束中和，以防止太空船带负电荷。

(a) Calculate the ratio of thrust F and the current I of the ion beam consisting of ions of mass m and charge ze, where z is a positive integer and e is the electronic charge. Express your answer in m, V, z and e. (4 points)
离子束由质量为 m、电荷为 ze 的离子组成，其中 z 是正整数，e 是电子电荷。试计算推力 F 与离子电流 I 之比。答案请以 m, V, z 和 e 表达。（4 分）

Current 电流: $I = Rze$ [1]

Using Newton's second law, 利用牛顿第二定律， $F = Rmv$ [1]

$$\frac{F}{I} = \frac{mv}{ze}$$

Using the conservation of energy, 利用能量守恒，

$$\frac{1}{2}mv^2 = zeV$$ [1]

$$\Rightarrow \quad v = \sqrt{\frac{2zeV}{m}} \quad \Rightarrow \quad \frac{F}{I} = \sqrt{\frac{2mV}{ze}}$$ [1]

(b) Calculate the ratio of thrust F and the power W spent in accelerating the ion beam. Express your answer in m, V, z and e. (2 points)
试计算推力 F 与加速离子束所耗功率 W 之比。答案请以 m, V, z 和 e 表达。（2 分）

Power 功率: $W = IV = RzeV$ [1]

$$\frac{F}{W} = \frac{mv}{zeV} = \sqrt{\frac{2m}{zeV}}$$ [1]

(c) To save power in space travel, should one prefer using light or heavy ions? Should one prefer using ions with single or multiple charges? Should one prefer using low or high accelerating voltages? (3 points)
为节省太空行程的功率，应该使用较轻抑较重的离子？应该使用单电荷离子抑多电荷离子？应该使用低加速电压抑高加速电压？（3 分）

Since F/W is proportional to $\sqrt{m}$, heavy ions are preferred.
由于 F/W 与 $\sqrt{m}$ 成正比，应该使用较重的离子。 [1]

Since F/W is proportional to $\sqrt{1/z}$, ions with single charge are preferred.
由于 F/W 与 $\sqrt{1/z}$ 成正比，应该使用单电荷离子。 [1]

Since F/W is proportional to $\sqrt{1/V}$, low accelerating voltages are preferred.
由于 F/W 与 $\sqrt{1/V}$ 成正比，应该使用低加速电压。 [1]

(d) A 10 kW electrostatic ion thruster using xenon atoms as propellant is designed. The accelerating voltage is 10 kV. Calculate the exhaust speed of the ions. Give your answer in km/s. Parameters: ionized xenon carries a single charge, atomic mass of xenon = 131, proton mass = 1.67×10^{-27} kg, electronic charge $e = 1.6 \times 10^{-19}$ C. (1 point)
一个 10 kW 的静电离子推进器的设计，使用氙原子作为推进剂。加速电压为 10 kV。试计算离子排出的速率。答案请以 km/s 为单位。参数：氙离子带单电荷，氙的原子质量 = 131，质子质量 = 1.67×10^{-27} kg，电子电荷 $e = 1.6 \times 10^{-19}$ C。（1 分）

$$v=\sqrt{\frac{2zeV}{m}}=\sqrt{\frac{(2)(1)(1.6\times10^{-19})(10\times10^{3})}{(131\times1.67\times10^{-27})}}=121\,\text{km/s}$$ [1]

(e) If the neutralizing electrode N of the thruster described in (d) is switched off, calculate the time taken by the body of the spacecraft to gain a voltage equal to the accelerating voltage; at that moment the thruster ceases to operate because the ions follow the thruster. Assume that the spacecraft is spherical and has a radius of 1 m. Parameters: ε_0 = 8.854 × 10^{-12} F/m. (2 points)
若在(d)中描述的推进器的中和电极 N 被关闭，试计算航天器身体上的电压变至与加速电压相等所需的时间；在那一刻因为离子不能离开推进器，将导致推进器停止操作。可假设航天器是球形的，半径为 1 m。参数：ε_0 = 8.854 × 10^{-12} F/m。（2 分）

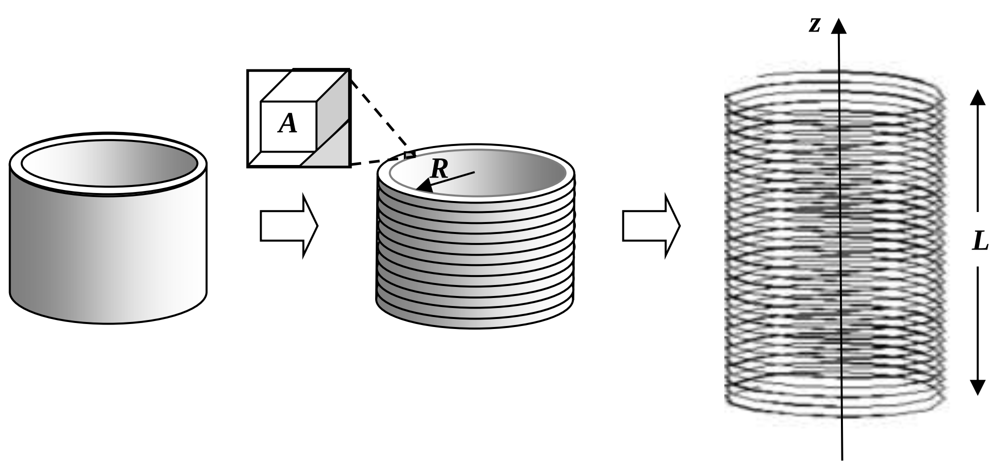
When the voltage has dropped to minus the accelerating voltage,
当电压降到加速电压的负值时，

$$V=\frac{Q}{4\pi\varepsilon_0 R}\quad\Rightarrow\quad Q=4\pi\varepsilon_0 RV=(4\pi)(8.854\times10^{-12})(1)(10\times10^{3})=1.11\times10^{-6}\text{ C}$$ [1]

$$I=\frac{W}{V}=\frac{10}{10}=1\text{ A}$$

Time for the voltage to drop 降低电压的时间: $t=\frac{Q}{I}=1.11\times10^{-6}\text{ s}=1.11\,\mu\text{s}$ [1]

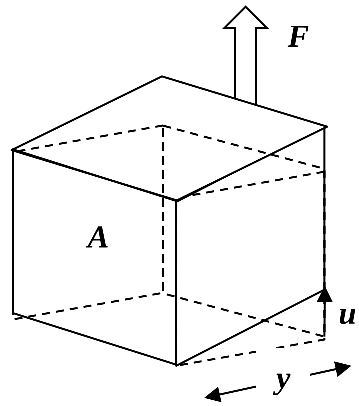
6. Slinky (10 points) 机灵鬼 （10 分）



The slinky is a spring first put on sale in 1940's, and soon became a popular toy. As shown in the figure, a slinky can be manufactured from a hollow metal cylinder of radius R by cutting it into a helical thin strip. The helix consists of N turns and has a cross sectional area A. Let ρ be the density of the metal.
机灵鬼弹簧首次于 1940 年代发售，很快便成为一种流行的玩具。如图所示，一個机灵鬼由半径为 R 的空心金属圆筒切割成螺旋形的薄带。螺旋线有 N 匝，其横截面面积为 A。设ρ是金属的密度。

(a) In this problem we assume that the deformation of a stretched slinky is mainly due to shear deformation. Let G be the shear modulus of the metal. What is the tension T in the slinky when it is stretched to a length L that is much greater than its original length?
在这问题中，我们假设机灵鬼被拉伸时的形变，主要是剪切形变。设 G 是金属的剪切模量。當把机灵鬼拉伸到长度 L 时（L 比机灵鬼原本的长度大得多），机灵鬼中的张力 T 是什么？

The shear modulus G of a solid is defined as $G = \frac{F/A}{u/y}$ where, as shown in the figure, F is the force acting on the vertical side of the solid with area A, y is the width of the solid, and u is the shear distortion of the solid. (2 points)
如图所示，固体的剪切模量 G 被定义为 $G = \frac{F/A}{u/y}$， 其中 F 是作用在固体侧面（面积为 A）的力，y 为固体的宽度，u 是固体的剪切形变。（2 分）



Considering the slinky as a sheared long strip, we have $y = 2\pi RN$ and $u = L$. Hence
把机灵鬼考虑成一条被剪切的长带，得 $y = 2\pi RN$ 和 $u = L$。所以

$$F = GA\frac{u}{y} = \frac{GAL}{2\pi RN}$$ [1]

Tension in the slinky 机灵鬼中的张力

$$T\sin\alpha = F$$

where α is the pitch angle given by $\sin\alpha = \frac{L}{2\pi RN}$.

其中 α 是斜角，由 $\sin\alpha = \frac{L}{2\pi RN}$ 给定。

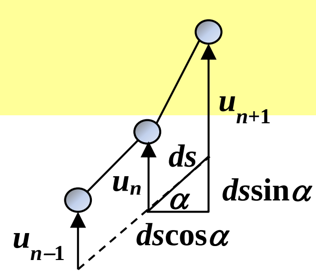
$$T = \frac{F}{\sin\alpha} = \left(\frac{GAL}{2\pi RN}\right)\left(\frac{2\pi RN}{L}\right) = GA$$ [1]

(b) To study how distortions propagate as a longitudinal wave in the slinky stretched to length L, we approximate the slinky by discrete particles separated by small distance ds connected by strings with tension T. Let $u_n(t)$ by the displacement of the n^{th} particle at time t. Derive the equation of motion of the particles. Neglect gravitational effects. (3 points)
为了研究形变如何以纵波在长度拉至 L 的机灵鬼上传播，我们将机灵鬼近似为一串离散的粒子，间距为 ds，由张力为 T 的繩子连接起来。设 $u_n(t)$ 是第 n 个粒子在时间 t 的位移。试推导粒子的运动方程。可忽略重力效应。（3 分）

$$\rho ds\ddot{u}_n = T\sin\theta_n - T\sin\theta_{n-1}$$

where θ_n is the inclination of the string from the n^{th} to $(n+1)^{th}$ particle.
其中 θ_n 是从第 n 个粒子到第 $n+1$ 个粒子的仰角。 [1]

$$\tan\theta_n = \frac{ds\sin\alpha + u_{n+1} - u_n}{ds\cos\alpha} \quad \Rightarrow$$



$$\sin\theta_n \approx \frac{ds\sin\alpha + u_{n+1} - u_n}{ds} = \sin\alpha + \frac{u_{n+1} - u_n}{ds} \qquad [1]$$

$$\sin\theta_{n-1} \approx \sin\alpha + \frac{u_n - u_{n-1}}{ds}$$

$$\rho ds\ddot{u}_n = T\sin\theta_n - T\sin\theta_{n-1} = \frac{GA}{ds}\left(u_{n+1} - 2u_n + u_{n-1}\right) \qquad [1]$$

(c) Show that $u_n(t) = C\sin(kz_n - \omega t)$ is a solution of the equation of motion, where z_n is the position of the n^{th} particle along the axis of the slinky. Find the relation between k and ω. Hence find the velocity of longitudinal wave propagation along the axis of the slinky. (5 points)
试证明 $u_n(t) = C\sin(kz_n - \omega t)$是运动方程的解，其中 z_n是沿机灵鬼轴线第 n 个粒子的位置，试找出 k 和ω之间的关系。由此推導沿机灵鬼轴线传播的纵波速度。（5 分）

Left hand side 左方: $\rho ds\ddot{u}_n = -\rho ds\omega^2 C\sin(kz_n - \omega t)$ [1]

Right hand side 右方: $\frac{GA}{ds}C\left[\sin(kz_{n+1} - \omega t) - 2\sin(kz_n - \omega t) + \sin(kz_{n-1} - \omega t)\right]$

$$= \frac{GA}{ds}C\left[2\sin\left(\frac{kz_{n+1} + kz_{n-1}}{2} - \omega t\right)\cos\left(\frac{kz_{n-1} - kz_{n+1}}{2}\right) - 2\sin(kz_n - \omega t)\right]$$

$$= -\frac{GA}{ds}2C\left[1 - \cos(kz_{n-1} - kz_{n+1})\right]\sin(kz_n - \omega t) \qquad [1]$$

Since 由于 $z_{n+1} - z_n = ds\sin\alpha$, $1 - \cos\left(\frac{kz_{n-1} - kz_{n+1}}{2}\right) = 1 - \cos(kds\sin\alpha) \approx \frac{1}{2}k^2ds^2\sin^2\alpha$, [1]

Right hand side 右方 $= -GAdsk^2\sin^2\alpha C\sin(kz_n - \omega t)$.

Comparing both sides, 比较两方，

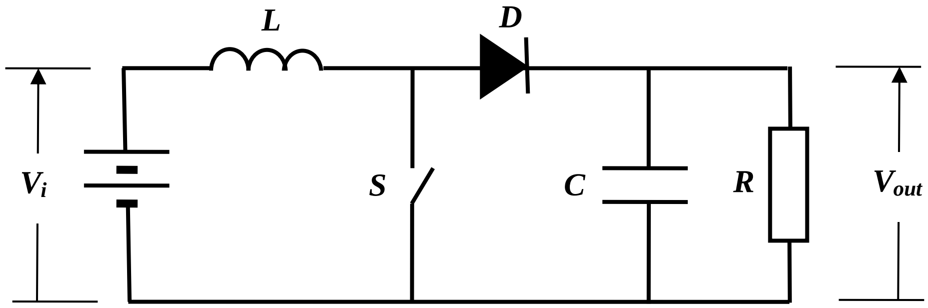
$$\rho\omega^2 = GA\sin^2\alpha k^2 \quad \Rightarrow \quad \omega^2 = \frac{GA\sin^2\alpha}{\rho}k^2 = \frac{GAL^2}{4\rho\pi^2R^2N^2}k^2 \qquad [1]$$

Wave velocity 波速: $c = \frac{\omega}{k} = \frac{L}{2\pi RN}\sqrt{\frac{GA}{\rho}}$ [1]

Part-2 (Total 2 Problems) 卷-2（共2 题）

1. DC Step-up Converter (25 points) 增压转换器 （25 分）

Modern electric and gasoline hybrid cars require high voltages to drive their motors from batteries of lower voltages. Alternating current (AC) voltages can be stepped up easily by using transformers, but direct current (DC) voltages require more sophisticated designs. In this problem we analyze the step-up converter circuit as shown in the following figure.
现代电力和汽油混合动力汽车需要从低电压的电池产生的高电压驱动马达。交流电（AC）电压可以很容易地通过使用变压器增强，但直流（DC）电压需要更复杂的设计才能做到這一點。在这个问题中，我们分析如下图所示的增压转换电路。



The circuit consists of an input voltage V_i, an inductor of inductance L, a capacitor of capacitance C, and a load of resistance R. D is a diode whose resistance is effectively zero when the electric potential is higher on the left end, and effectively infinite when the electric potential is lower on the left end.
该电路包括一个输入电压 V_i，一個电感为 L 的电感器，一個电容为 C 的电容器，和一個电阻为 R 的负载。D 是一个二极管，當左端电势高時，二极管的有效电阻是零，當左端电势低時，二极管的有效电阻是無限大。

S is a switch operated by an electronic circuit not shown in the figure. It switches on and off periodically at a rather high frequency. Each period consists of an on-state and an off-state. During the on-state, it is switched on for a time t_1, and during the off-state, it is switched off for a time t_0.
S 是一個由图中未显示的电子电路所控制的开关。它以一个相当高的频率作周期性地开关。每个周期包括一个导通状态和一個关断状态。在导通状态时，它被接通的时间为 t_1，在关断状态时，它处于关闭状态的时间为 t_0。

(a) Consider the initial condition that the current in the circuit is 0 and the capacitor is uncharged. At $t = 0$, switch S is closed. Calculate the current through the inductor at $t = t_1$. (2 points)
考虑初始状态時电路中的电流为 0，电容器是不带电的。在 $t = 0$ 时，开关 S 闭合。试计算在 $t = t_1$ 時通过电感器的电流。（2 分）

When switch S is closed, the diode, capacitor and resistor can be ignored. Hence
当开关 S 关闭时，二极管，电容器和负载可忽略不计。故

$$V_i - L\frac{dI}{dt} = 0 \qquad [1]$$

$$\Rightarrow \quad I = \int_0^{t_1} \frac{V_i}{L} dt = \frac{V_i}{L} t_1 \qquad [1]$$

(b) At $t = t_1$, switch S is open. Calculate the current through the inductor at time t during the off-state ($t_1 < t < t_1 + t_0$). You may assume that the load resistance R is so large that the current it draws is negligible. (6 points)
在 $t = t_1$ 时，开关 S 断开。试计算在关断状态中时间为 t（$t_1 < t < t_1 + t_0$）时，通过电感器的电流。你可以假设负载电阻 R 很大，通過它的电流可以忽略不計。（6 分）

When switch S is open, the circuit consists of the battery, inductor, diode and capacitor (the resistor is ignored). Assuming that the diode resistance is 0,
当开关 S 断开时，电路包括电池，电感器，二极管和电容器（负载可忽略）。假设二极管电阻为 0，

$$V_i - L\frac{dI}{dt} - \frac{q}{C} = 0$$ [1]

$$I = \frac{dq}{dt} \quad \Rightarrow \quad \frac{d^2q}{dt^2} + \frac{q}{LC} = \frac{V_i}{L} \quad \Rightarrow \quad \frac{d^2q}{dt^2} + \frac{1}{LC}(q - CV_i) = 0$$ [1]

The motion of $q - CV_i$ is a simple harmonic motion with angular frequency $\omega = \frac{1}{\sqrt{LC}}$.

$q - CV_i$ 的变化是简谐运动，其角频率为 $\omega = \frac{1}{\sqrt{LC}}$ [1]

Let 设 $q = CV_i + A\sin\omega(t - t_1) + B\cos\omega(t - t_1) \Rightarrow I = \omega A\cos\omega(t - t_1) - \omega B\sin\omega(t - t_1)$.

The initial condition is $q = 0$ and $I = V_i t_1/L$ at $t = t_1$.

初始条件为 $q = 0$ and $I = V_i t_1/L$ at $t = t_1$.

$B = -CV_i$ [1]

$A = \frac{V_i t_1}{\omega L}$ [1]

$$I = \frac{V_i t_1}{L}\cos\omega(t - t_1) + \omega CV_i \sin\omega(t - t_1)$$ [1]

(c) When the device continues to operate, we will consider the high frequency limit in which $t_1 + t_0 << \sqrt{LC}$ in the rest of the problem. In this regime, it is sufficient to keep terms up to first order of t_0 and t_1. Find the relation between the current through the inductor at the end of the $(n - 1)^{\text{th}}$ off-state and that of the n^{th} on-state, denoted as $I_0(n - 1)$ and $I_1(n)$ respectively. (2 points)

当电路持续工作時，我们将在下面的问题中，考虑高频极限 $t_1 + t_0 << \sqrt{LC}$ 。在此条件下，只需考虑 t_0 和 t_1 的一阶项。试找出通过电感器的电流在第 $n - 1$ 次关断状态结束时（定义为 $I_0(n - 1)$）与在第 n 次导通状态結束时（定义为 $I_1(n)$）之间的关系。（2 分）

During the next on-state, switch S is closed. Hence

在下一次导通状态，开关 S 闭合。故

$$V_i - L\frac{dI}{dt} = 0$$ [1]

$$\Rightarrow \quad I_1(n) = I_0(n-1) + \int_0^{t_1}\frac{V_i}{L}dt = I_0(n-1) + \frac{V_i}{L}t_1$$ [1]

(d) By including the load in the circuit during the n^{th} on-state, find the relation between the voltage across the capacitor at the end of the $(n - 1)^{\text{th}}$ off-state and that of the n^{th} on-state, denoted as $V_0(n - 1)$ and $V_1(n)$ respectively. (2 points)

试在第 n 次导通状态期间考虑把负载包括在电路中，从而找出电容器两端的电压在第 $n - 1$ 次关断状态結束時（定義為 $V_0(n-1)$）与在第 n 次导通状态結束时（定義為 $V_1(n)$）之间的关系。（2 分）

During the n^{th} on-state, the capacitor discharges and produces the current through the load.

在第 n 次导通状态中，电容器放电并产生通过负载的电流。

$$\frac{q}{C} - IR = 0$$ [1]

$$I=-\frac{dq}{dt} \quad \Rightarrow \quad \frac{dq}{dt}+\frac{q}{RC}=0 \quad \Rightarrow \quad q=q_0(n-1)e^{-\frac{t}{RC}}$$

$$V_1(n)=V_0(n-1)e^{-\frac{t_1}{RC}}$$

$$\approx V_0(n-1)-\frac{V_0(n-1)}{RC}t_1$$ [1]

(e) At the end of the n^{th} on-state, the current through the inductor is $I_1(n)$, and the voltage across the capacitor is $V_1(n)$. Calculate the current $I_0(n)$ through the inductor and the voltage $V_0(n)$ across the capacitor at the end of the immediately following off-state. (5 points)
在第 n 次导通状态結束时，通过电感器的电流为 $I_1(n)$，在电容器两端的电压是 $V_1(n)$。试計算在紧随的关断状态結束时通过电感器的电流 $I_0(n)$以及电容器两端的电压 $V_0(n)$。 （5 分）

During the immediately following off-state, switch S is open. Hence
在紧随的关断状态結束时，开关 S 断开。故

$$V_i-L\frac{dI}{dt}-\frac{q}{C}=0$$ [1]

The solution is the same as that of (b), but the initial condition is modified.
方程解與(b)相同，但初始条件不同。

Let $q=CV_i+A\sin(\omega t)+B\cos(\omega t)$, where the time is measured from the end of the n^{th} on-state.
设 $q=CV_i+A\sin(\omega t)+B\cos(\omega t)$, 其中时间从第 n 次导通状态结束时开始计算。

$I=\omega A\cos(\omega t)-\omega B\sin(\omega t)$.

The initial condition is 初始条件为 $q = CV_1(n)$ and $I = I_1(n)$ at $t = 0$.

$$B=CV_1(n)-CV_i$$ [1]

$$A=\frac{I_1(n)}{\omega}$$ [1]

$$I_0(n)=I_1(n)\cos(\omega t_0)-\omega\left[CV_1(n)-CV_i\right]\sin(\omega t_0)$$

$$\approx I_1(n)-\left[CV_1(n)-CV_i\right]\omega^2 t_0=I_1(n)-\frac{V_1(n)-V_i}{L}t_0$$ [1]

$$V_0(n)=\frac{q(t_0)}{C}=V_i+\frac{I_1(n)}{\omega C}\sin(\omega t_0)+\left[V_1(n)-V_i\right]\cos(\omega t_0)$$

$$\approx V_1(n)+\frac{I_1(n)}{C}t_0$$ [1]

(f) When the device reaches the steady state, calculate the step-up voltage ratio V_{out}/V_i to the lowest order. How should we set t_1 and t_0 to raise the ratio? (3 points)
当电路达到稳定状态时，试計算增压电压比率 V_{out}/V_i，以最低阶解答即可。为了提高這个比率，该如何设置 t_1 和 t_0? （3 分）

When the device reaches the steady state, 当電路达到稳定状态，
$V_1(n-1) = V_1(n)$ and $I_1(n-1) = I_1(n)$.

$$I_1=I_0+\frac{V_i}{L}t_1$$

$$I_0\approx I_1-\frac{V_1-V_i}{L}t_0$$ [1]

Eliminating I_1 and I_0, 消去 I_1 和 I_0， $V_1 \approx V_i \frac{t_1 + t_0}{t_0}$.

To the lowest order, 保留最低阶， $V_0 \approx V_1$.

Hence 所以 $\frac{V_{out}}{V_i} \approx \frac{t_1 + t_0}{t_0}$. [1]

To raise the step-up voltage ratio, t_1 should be set longer than t_0.
为了提高增压电压比率，t_1 应该比 t_0 長。 [1]

(g) Calculate the on-state current through the inductor at the steady state. Explain the physical meaning of the result. (3 points)
试计算当电路达到稳定状态后，导通状态时通过电感器的电流。解释结果的物理意义。（3 分）

$V_1 \approx V_0\left(1 - \frac{t_1}{RC}\right)$

$V_0 \approx V_1 + \frac{I_1}{C} t_0$ [1]

Eliminating V_0, 消去 V_0, $I_1 \approx \frac{V_1}{R}\frac{t_1}{t_0}$ [1]

Physical meaning: $I_1 t_0$ is the charge stored in the capacitor during the off-state. V_1/R is the current flowing out of the capacitor during the on-state, and $V_1 t_1/R$ is the charge drained from the capacitor during the on-state. The two quantities are the same due to charge conservation.
物理意义：$I_1 t_0$ 是在关断状态存储在电容器中的电荷。V_1/R 是在导通状态下从电容器流出的电流，$V_1 t_1/R$ 是在导通状态下从电容流出的电荷。由于电荷守恒，这两个量是相同的。 [1]

(h) Explain the importance of the diode in producing the step-up voltage. (1 point)
试解释二极管在产生增压电压中的重要性。（1 分）

The diode prevents the capacitor to discharge through shorted circuit during the on-state, and through the battery and the inductor during the off-state, so that the voltage across the capacitor can build up to a high value.
在导通状态下，二极管可防止电容器通过短路放电。在关断状态下，二极管可防止电容器通过电池和电感器放电。这样电容器两端的电压可以升高。 [1]

(i) Estimate the time taken to reach the steady state. Use only the variables t_1, t_0, L, C, R to express your result. (1 point)
试估计达到稳定状态所需的时间。只可使用变量 t_1, t_0, L, C, R 表达你的结果。（1 分）

The current increases by $V_i t_1/L$ per period. The steady state current is $V_1 t_1/Rt_0$. Hence we estimate the number of periods to reach the steady state is $\left(\frac{V_1 t_1}{Rt_0}\right)\left(\frac{V_i t_1}{L}\right)^{-1} = \left(\frac{V_1}{V_i}\right)\left(\frac{L}{Rt_0}\right)$. The estimated time is $\left(\frac{V_1}{V_i}\right)\left(\frac{L}{Rt_0}\right)(t_1 + t_0) = \left(\frac{t_1 + t_0}{t_0}\right)^2\left(\frac{L}{R}\right)$.

电流在每個周期增加 $V_i t_1/L$。稳态电流是 $V_1 t_1/Rt_0$。因此我们估计达到稳定状态的周期数是 $\left(\frac{V_1 t_1}{Rt_0}\right)\left(\frac{V_i t_1}{L}\right)^{-1} = \left(\frac{V_1}{V_i}\right)\left(\frac{L}{Rt_0}\right)$，所需時間約為 $\left(\frac{V_1}{V_i}\right)\left(\frac{L}{Rt_0}\right)(t_1 + t_0) = \left(\frac{t_1 + t_0}{t_0}\right)^2\left(\frac{L}{R}\right)$。 [1]

2. White Dwarf (25 points) 白矮星 （25 分）

At the end of lives of stars with comparable masses as the Sun, the gravitational force compresses the star inward to form white dwarfs, and is eventually balanced by the quantum mechanical pressure of the electrons (known as the degeneracy pressure). This determines the size of the white dwarfs, which is comparable to that of the Earth. In this problem we analyze the size of white dwarfs.

当质量与太阳相近的恒星终结时，引力会使恒星向内坍塌形成白矮星。引力最终与电子气体的量子效应造成的压力（称为简并压）平衡。这决定了白矮星的大小与地球近似。本体旨在分析白矮星的大小。

(a) First consider an electron of mass m_e confined in a one-dimensional box of length L. Its kinetic energy is given by $E=\frac{p^2}{2m_e}$, where p is the momentum of the electron. In quantum theory, the electrons are described by waves whose wavelengths λ determine the momenta by the de Broglie relation $p=\frac{h}{\lambda}$. Only standing waves with nodal points at the wall of the box give rise to the allowed electronic states of the electrons. This enables us to calculate the energy of the n^{th} state as $E_n = E_1 n^2$. Derive the expression E_1. (3 points)

首先考虑质量为 m_e 的电子局限在长度为 L 的一维盒子中。其动能为 $E=\frac{p^2}{2m_e}$，p 是电子的动量。在量子理论中，电子可以用波来描述，其波长透过德布罗意关系决定动量 $p=\frac{h}{\lambda}$。只有电子波形成驻波的节点处于盒子两端时，才是允许的电子态。这使我们能够计算的第 n 个电子态的能量为 $E_n = E_1 n^2$。试导出 E_1 的表达式。 （3 分）

The n^{th} standing wave in the box is given by 盒中第 n 个驻波满足 $\frac{n\lambda}{2}=L \quad \Rightarrow \quad \lambda=\frac{2L}{n}$. [1]

The momentum of the n^{th} electronic state is 第 n 个电子态的动量为

$$p=\frac{h}{\lambda} \quad \Rightarrow \quad p=\frac{h}{2L}n. \qquad [1]$$

The energy of the n^{th} electronic state is 第 n 个电子态的能量为 $E=\frac{h^2}{8m_eL^2}n^2$.

Hence 所以 $E_1=\frac{h^2}{8m_eL^2}$ [1]

(b) To simplify the picture, we consider the white dwarf as a three-dimensional cubic box with volume V. The energy of an electronic state in the box is $E=E_1(n_x^2+n_y^2+n_z^2)$, where n_x, n_y, n_z are positive integers. Calculate the total number of electronic states with energy below the maximum energy E_{max}. Assume that E_{max} is much greater than E_1. (2 points)

作为简化模型，我们把白矮星考虑成一个体积为 V 的三维立方盒子。电子态的能量为 $E=E_1(n_x^2+n_y^2+n_z^2)$，其中 n_x, n_y, n_z 是正整数。试计算低于最大能量 E_{max} 的电子态的总数。假定 E_{max} 远大于 E_1。（2 分）

We have to find the number of lattice points satisfying $n_x^2+n_y^2+n_z^2 \le E_{max}/E_1$. In the three-dimensional space, these points are enclosed in the first octant of the sphere of radius $\sqrt{E_{max}/E_1}$.
满足 $n_x^2+n_y^2+n_z^2 \le E_{max}/E_1$ 的正整数组(n_x, n_y, n_z)的数目即三维空间第一卦限中，半径为 $\sqrt{E_{max}/E_1}$ 的球体内的整格点数。 [1]

Since the volume enclosing a point is 1, the number of lattice points in one-eighth of a sphere is
由于每个格点体积为 1，八分之一个球体包含格点数为

$$N_{\text{state}} = \frac{1}{8}\frac{4}{3}\pi\left(\frac{E_{max}}{E_1}\right)^{\frac{3}{2}} = \frac{\pi V}{6h^3}\left(8m_e E_{max}\right)^{\frac{3}{2}}. \quad [1]$$

(c) Suppose there are N protons and N electrons in the white dwarf. Due to the famous Pauli exclusion principle in quantum mechanics, each electronic state can only accommodate 2 electrons. The electrons will fill up the electronic states from low to high energy up to a maximum energy called the Fermi energy E_F. Calculate E_F. (2 points)
假设白矮星内有 N 个质子和 N 个电子。根据量子力学中著名的泡利不相容原理，每个电子态只能容纳 2 个电子。电子会按能量从低到高填满所有可能的电子态，直到能量达到最大能量 E_F，E_F 称为费米能。试计算 E_F。（2 分）

$$N = 2N_{\text{state}} = \frac{\pi V}{3h^3}\left(8m_e E_F\right)^{\frac{3}{2}} \quad [1]$$

$$\Rightarrow \quad E_F = \frac{h^2}{8m_e}\left(\frac{3N}{\pi V}\right)^{\frac{2}{3}} \quad [1]$$

(d) Calculation shows that the average energy per electron is $3E_F/5$. Considering the electrons as a gas, what is the pressure of the electron gas? Is the pressure inward or outward? (4 points)
计算显示，平均每个电子的能量为 $3E_F/5$。将电子作为气体，电子气的压强是多少？压力是向内还是向外？（4 分）

Total kinetic energy of the electron gas 电子气的总动能 $E = \frac{3}{5}NE_F = \frac{3h^2}{40m_e}\left(\frac{3}{\pi V}\right)^{\frac{2}{3}} N^{\frac{5}{3}}$ [1]

Using the first law of thermodynamics, 利用热力学第一定律， $dE = -PdV$.

Therefore, 所以， $P = -\frac{dE}{dV}$ [1]

$$= \frac{h^2}{20m_e}\left(\frac{3}{\pi}\right)^{\frac{2}{3}}\left(\frac{N}{V}\right)^{\frac{5}{3}} \quad [1]$$

The pressure is outward. 压力向外。 [1]

(e) Compare the degeneracy pressure due to protons with that due to electrons. (1 point)
比较电子气的简并压与质子气的简并压。（1 分）

Since the pressure is inversely proportional to the mass of the particle, the degeneracy pressure of protons is much less than that of electrons.
由于压力与粒子的质量成反比，质子的简并压力比电子的少得多。 [1]

(f) The gravitational potential energy is dominated by protons and neutrons. Let m_p be the mass of a proton or a neutron. Assume that the number of protons and neutrons are the same, and the mass density is approximately constant inside the star. Calculate the gravitational potential energy of the star of radius R. (4 points)
引力势能主要由质子和中子贡献。质子或中子的质量为 m_p。设质子和中子的数目相同，并且恒星内质量密度近似为常数。试计算半径为 R 的恒星的引力势能。（4 分）

Mass of the star of radius r 半径为 r 的恒星质量为 $2N\left(\frac{r^3}{R^3}\right)m_p$。 [1]

Suppose its radius increases by a thin shell of thickness dr. Mass of the thin shell:
若增加一厚度为 dr 的薄球壳，球壳质量:

$$dm = 2N\left(\frac{4\pi r^2 dr}{4\pi R^3/3}\right)m_p = \frac{6Nm_p r^2}{R^3}dr$$ [1]

Change in potential energy 引力势能变化

$$dU = -G2N\left(\frac{r^3}{R^3}\right)m_p\frac{dm}{r} = -\frac{12GN^2m_p^2r^4}{R^6}dr$$ [1]

Gravitational potential energy 总引力势能

$$U = -\int_0^R \frac{12GN^2m_p^2r^4}{R^6}dr = -\frac{12GN^2m_p^2}{5R}$$ [1]

(g) Derive the expression of the radius of the white dwarf. Does the radius increase or decrease with increasing mass of the white dwarf? (4 points)
试推导白矮星半径的表达式。若白矮星质量增加，半径是增加还是减少？ （4 分）

Total energy of the white dwarf 白矮星的总能量

$$E_{\text{tot}} = \frac{3h^2}{40m_e}\left(\frac{3}{\pi V}\right)^{\frac{2}{3}}N^{\frac{5}{3}} - \frac{12GN^2m_p^2}{5R} = \frac{3h^2}{40m_e}\left(\frac{9}{4\pi^2}\right)^{\frac{2}{3}}\frac{N^{\frac{5}{3}}}{R^2} - \frac{12GN^2m_p^2}{5R}$$ [1]

Minimizing the total energy, 平衡时总能量最小，

$$\frac{dE_{\text{tot}}}{dR} = -\frac{3h^2}{20m_e}\left(\frac{9}{4\pi^2}\right)^{\frac{2}{3}}\frac{N^{\frac{5}{3}}}{R^3} + \frac{12GN^2m_p^2}{5R^2} = 0$$

[1]

$$R = \frac{h^2}{16Gm_p^2m_e}\left(\frac{9}{4\pi^2}\right)^{\frac{2}{3}}\left(\frac{1}{N}\right)^{\frac{1}{3}}$$ [1]

Since R is proportional to $N^{-1/3}$, the radius decreases with increasing mass of the white dwarf.
由于 R 与 $N^{-1/3}$ 成正比，白矮星质量增加则半径减小。 [1]

(h) Calculate the radius of the white dwarf with the same mass as the Sun. Give your answer in multiples of Earth's radius. You are given the following parameters: (2 points)
试计算质量与太阳相同的白矮星半径。答案请以地球半径为单位。可用以下参数：（2 分）

h = Planck's constant 普朗克常数 = 6.626×10^{-34} Js
G = gravitational constant 万有引力常数 = 6.67×10^{-11} Nm^2/kg^2
m_p = mass of a proton or neutron 质子或中子质量 = 1.67×10^{-27} kg

m_e = mass of an electron 电子质量 = 9.11×10^{-31} kg
m_{Sun} = mass of Sun 太阳质量 = 1.99×10^{30} kg
R_E =radius of Earth 地球半径 = 6380 km

Number of protons in the star 白矮星内质子数目 $N = \frac{m_{Sun}}{2m_p}$ [1]

$$R = \frac{(6.626\times10^{-34})^2}{(16)(6.67\times10^{-11})(1.67\times10^{-27})^2(9.11\times10^{-31})}\left(\frac{9}{4\pi^2}\right)^{\frac{2}{3}}\left(\frac{(2)(1.67\times10^{-27})}{1.99\times10^{30}}\right)^{\frac{1}{3}}$$

=7181 km = $1.13R_E$ [1]

(i) Estimate the mass of the white dwarf when the velocity of electrons becomes comparable to the velocity of light $c = 3 \times 10^8$ m/s. Give your answer in multiples of solar mass. What will happen to the white dwarf? (3 points)
当电子速度接近光速 $c = 3 \times 10^8$ m/s 时，试估计白矮星的质量。请以太阳质量为单位。白矮星将有什么发生？（3 分）

We estimate the velocity of electrons in the white dwarf by 白矮星内电子的速度可估计为

$$v = \frac{p}{m_e} = \frac{1}{m_e}\sqrt{p_x^2 + p_y^2 + p_z^2} = \frac{h}{2m_e L}\sqrt{n_x^2 + n_y^2 + n_z^2}$$

When 当 $v = c$, $\sqrt{n_x^2 + n_y^2 + n_z^2} = \frac{2m_e Lc}{h} = \frac{2m_e c}{h}V^{\frac{1}{3}}$. [1]

The number of states enclosed by a sphere with this radius 在此球体中的电子状态数

$$N_{\text{state}} = \frac{1}{8}\frac{4}{3}\pi\left(\frac{2m_e c}{h}\right)^3 V = \frac{4\pi m_e^3 c^3}{3h^3}V$$

The number of protons in the white dwarf 白矮星中的质子数

$$N = 2N_{\text{state}} = \frac{8\pi m_e^3 c^3}{3h^3}\left(\frac{4}{3}\pi R^3\right) = \frac{32\pi^2 m_e^3 c^3}{9h^3}R^3$$

Using the result from (g), 从（g）中的结果可得

$$N = \frac{9c^3h^3}{2048\pi^2 G^3 m_p^6}\left(\frac{1}{N}\right) \quad \Rightarrow \quad N = \sqrt{\frac{9c^3h^3}{2048\pi^2 G^3 m_p^6}} = \frac{3}{16\pi m_p^3}\left(\frac{hc}{2G}\right)^{\frac{3}{2}}$$

Mass of the white dwarf 白矮星的质量为 $2Nm_p = \frac{3}{8\pi m_p^2}\left(\frac{hc}{2G}\right)^{\frac{3}{2}}$

$$= \frac{3}{8\pi(1.67\times10^{-27})^2}\left(\frac{(6.626\times10^{-34})(3\times10^8)}{2(6.67\times10^{-11})}\right)^{\frac{3}{2}} = 2.46\times10^{30}\text{ kg} = 1.2m_{\text{Sun}}$$ [1]

When the mass of the white dwarf reaches this limit, it will no longer exist. 当白矮星的质量达到此阈值时，将不再存在。
Remark: This answer is comparable to the Chandrasekhar limit of 1.4 m_{Sun}. Beyond this limit, it will collapse into a neutron star.
备注：此答案接近钱德拉塞卡极限 1.4 m_{Sun}。超过此限，星体会坍塌形成中子星。 [1]