Pan Pearl River Delta Physics Olympiad 2015

2015 年泛珠三角及中华名校物理奥林匹克邀请赛

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香港科技大学高等研究院赞助

Part-1 (Total 5 Problems) 卷-1（共5题）

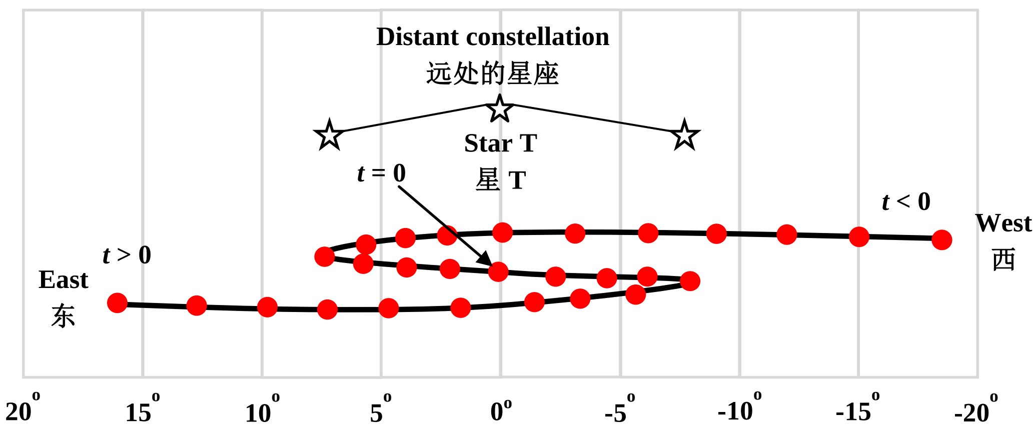
(9:00 am – 12:00 pm, 25 February, 2015)

Numerical answers should be given to 3 significant figures.数字答案请给三位有效数字。

1. Retrograde Motion of Mars (9 points) 火星的逆行运动 （9 分）

In the history of astronomy, the phenomenon of the retrograde motion played an important role. Suppose we observe the position of Mars at midnight every night for many nights. Using distant stars and constellations as the background, we will find that Mars moves from West to East most of the time. However, there are periods of time that Mars is observed to move in opposite direction, as shown in the figure. The orbital period of Mars is 1.88 y. Assume that the orbits of Earth and Mars are circular, and the tilting of Earth's axis can be ignored.

在天文史上，行星的逆行运动扮演了重要的角色。假设我们连续多个晚上在午夜观察火星的位置。若以远处的星体和星座为背景，我们会发现大部分时间火星是从西到东运动，但也有些时段是逆向运动，如图所示。火星的轨道周期是 1.88 年。假设地球和火星的轨道都是圆的，地轴的倾斜可略。

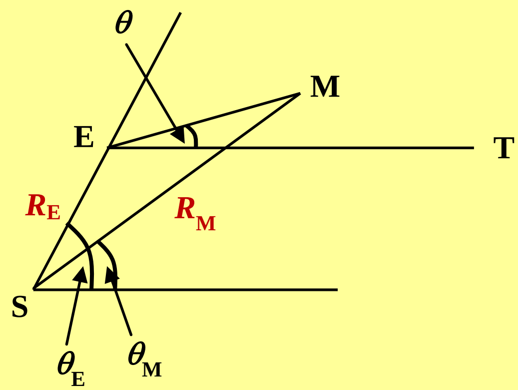


(a) What is the orbital radius R_M of Mars? Give your answer in AU (Astronomical Units, 1 AU is the average distance between Sun and Earth.) (1 points)
试求火星的轨道半径 R_M。答案请以 AU 为单位。(1 AU 是太阳与地球的平均距离。)（1 分）

Using Kepler's Law, 按开普勒定律， $R_M = R_E\left(\frac{T_M}{T_E}\right)^{\frac{2}{3}} = 1\left(\frac{1.88}{1}\right)^{\frac{2}{3}} = 1.5233\text{ AU} \approx 1.52\text{ AU}$

(b) At $t = 0$, Sun, Earth and Mars lie on a straight line. Sketch a figure indicating the positions of Sun, Earth, Mars, and star T when $t > 0$. Label them by letters S, E, M, and T respectively. Mark the angular displacements θ_E and θ_M of Earth and Mars respectively (starting from t =

0), and the angle θ that gives the angular position of Mars as observed from Earth using distant stars and constellations as the background. (2 points)
在 t = 0 时，太阳、地球、火星成一直线。试作一草图，显示在 t > 0 时，太阳、地球、火星和星 T 的位置，以 S、E、M 和 T 标示。在图上标示地球和火星的角位移分别为 θ_E 和 θ_M（自 t = 0 开始），和地球观察火星的角位置θ（以远处的星体和星座为背景）。（2 分）



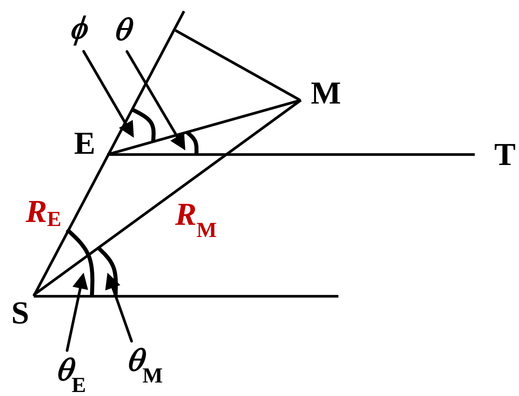
(c) Derive an expression for the angular position θ of Mars at time t. Express your answer in terms R_E, R_M, ω_E, ω_M and t, where ω_E and ω_M are the orbital angular velocity of Earth and Mars respectively. (4 points)
试推导火星在时间 t 时的角位置θ。答案请以 R_E, R_M, ω_E, ω_M 和 t 表示，其中ω_E 和ω_M 分别为地球与火星的角速度。（3 分）

$\theta_E = \omega_E t$, $\theta_M = \omega_M t$

In triangle SEM, we need to find the exterior angle at E. By constructing a perpendicular line from M to SE, this angle is
在三角形 SEM 里，需找出角 E 的外角。从点 M 作一线垂直于 SE，可见这角为

$$\phi = \tan^{-1}\left(\frac{R_M \sin(\theta_E - \theta_M)}{R_M \cos(\theta_E - \theta_M) - R_E}\right).$$

$$\theta = \omega_E t - \tan^{-1}\left(\frac{R_M \sin(\omega_E t - \omega_M t)}{R_M \cos(\omega_E t - \omega_M t) - R_E}\right)$$



(d) Calculate the angular position θ of Mars at t = 0.1 y, 0.2 y and 0.3 y. Give your answer in degrees. (3 points)
试计算火星在 t = 0.1 年, 0.2 年和 0.3 年时的角位置θ。答案请以度数表示。（3 分）

At $t = 0.1$, $\theta = 2\pi(0.1) - \tan^{-1}\left(\frac{1.5233\sin[2\pi(0.1) - 2\pi(0.1)/1.88]}{1.5233\cos[2\pi(0.1) - 2\pi(0.1)/1.88] - 1}\right) = -7.963^{\circ} \approx -8.00^{\circ}$

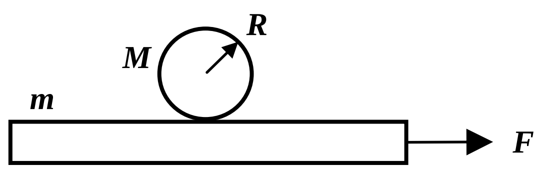
At $t = 0.2$, $\theta = 2\pi(0.2) - \tan^{-1}\left(\frac{1.5233\sin[2\pi(0.2) - 2\pi(0.2)/1.88]}{1.5233\cos[2\pi(0.2) - 2\pi(0.2)/1.88] - 1}\right) = -0.4538^{\circ} \approx -0.454^{\circ}$

At $t = 0.3$, $\theta = 2\pi(0.3) - \tan^{-1}\left(\frac{1.5233\sin[2\pi(0.3) - 2\pi(0.3)/1.88]}{1.5233\cos[2\pi(0.3) - 2\pi(0.3)/1.88] - 1}\right) = -16.43^{\circ} \approx 16.4^{\circ}$

2. Rolling Ball on a Racket (10 points) 球拍滚球（10 分）

As shown in the figure, a hollow spherical ball of mass M and radius R is placed on a racket of mass m. The racket has a flat surface with coefficient of static friction μ_s and coefficient of kinetic friction μ_k and is held horizontally.

如图所示，一个质量为 M，半径为 R 的空心园球被放置在质量为 m 的球拍上。球拍具有一个平坦的表面，其静摩擦系数为μ_s，动摩擦系数为μ_k，并且被保持在水平位置。



(a) The racket is driven horizontally by a periodic force $F(t) = F_0 \cos\omega_0 t$, with the ball remaining non-slipping. Calculate the maximum velocities of the oscillations of the racket and the ball, denoted as u_x and u_y respectively. (The moment of inertia of a hollow sphere of mass M and radius R is $I = 2MR^2/3$.) (5 points)

球拍被周期性的力 $F(t) = F_0 \cos\omega_0 t$ 沿水平方向驱动，园球维持在不滑动的状态。试计算球拍与球振动时的最大速度，分别表示为 u_x 和 u_y。（质量为 M，半径为 R 的空心球体的转动惯量为 $I = 2MR^2/3$。） (5 分)

Let x and y be the displacements of the racket and the ball respectively. Let θ be the angular displacement of the ball (counted in the direction of x at the contact point with the racket). Let f be the frictional force between the racket and the ball. Applying Newton's law,

令 x 和 y 分别为球拍与球的位移。令θ为球的角位移（方向按照在与球拍的接触点沿位移 x 方向计算）。设 f 是球拍与球之间的摩擦力。运用牛顿定律，

$$m\ddot{x} = F - f \qquad (1)$$

$$M\ddot{y} = f \qquad (2)$$

$$\frac{2}{3}MR^2\ddot{\theta} = fR \qquad (3)$$

Condition for no slipping: 不滑动的条件：$\theta = \frac{x - y}{R}$ (4)

From Eqs. (3) and (4), 从方程式（3）和（4），$\frac{2}{3}M(\ddot{x} - \ddot{y}) = f$ (5)

Combining with Eq. (2), 结合方程式（2）：$\ddot{y} = \frac{2}{5}\ddot{x}$ and $f = \frac{2}{5}M\ddot{x}$ (6)

Substituting into Eq. (1), 代入方程式（1），$\left(m + \frac{2}{5}M\right)\ddot{x} = F_0 \cos\omega_0 t$

Solution: 解：$x = -\frac{5F_0 \cos\omega_0 t}{(2M + 5m)\omega_0^2}$, $y = -\frac{2F_0 \cos\omega_0 t}{(2M + 5m)\omega_0^2}$, $\theta = -\frac{3F_0 \cos\omega_0 t}{(2M + 5m)\omega_0^2 R}$,

$\dot{x} = \dfrac{5F_0 \sin \omega_0 t}{(2M+5m)\omega_0}$, $\dot{y} = \dfrac{2F_0 \sin \omega_0 t}{(2M+5m)\omega_0}$, $\dot{\theta} = \dfrac{3F_0 \sin \omega_0 t}{(2M+5m)\omega_0 R}$.

Hence 因此 $u_x = \dfrac{5F_0}{(2M+5m)\omega_0}$, $u_y = \dfrac{2F_0}{(2M+5m)\omega_0}$.

(b) At the moment the racket is oscillating at its maximum velocity, its motion is brought to rest abruptly by an external force much stronger than the limiting frictional force between the racket and the ball in a very short duration of time. What is the final velocity of the ball? If the final velocity of the ball is 0, what is the displacement of the ball? (5 points)
在球拍振动至最大速度的一刻，其运动突然被外力煞停，这外力比球拍与球之间的极限摩擦力强得多，作用的时间也很短。问球的最终速度是多少？若球的最终速度为 0，其位移是多少？（5 分）

The impulse acting on the racket is given by the external force multiplied by the time duration of the force, whereas the impulse acting on the ball is given by the limiting frictional force multiplied by the duration. Hence the impulse acting on the ball is negligible. Hence when the racket stops moving, the ball continues to move with the velocity u_y and angular velocity $\omega \equiv \dfrac{3F_0}{(2M+5m)\omega_0 R} = \dfrac{3u_y}{2R}$. Since $u_y \neq -R\omega$, the ball will slide until it finally rolls. Applying Newton's law,
作用在球拍的冲量是外力乘以力作用的时间，而作用在球上的冲量是极限摩擦力乘以力作用的时间。因此，作用在球上的冲量可以忽略不计。因此，当球拍停止移动时，球继续以速度 u_y 和角速度 $\omega \equiv \dfrac{3F_0}{(2M+5m)\omega_0 R} = \dfrac{3u_y}{2R}$ 移动。因 $u_y \neq -R\omega$，球会滑动，直到它最终滚动。运用牛顿定律，

$$M\ddot{y} = -\mu_k Mg \quad \Rightarrow \quad \dot{y} = u_y - \mu_k gt$$

$$\frac{2}{3}MR^2\ddot{\theta} = -fR \quad \Rightarrow \quad R\dot{\theta} = \frac{3}{2}u_y - \frac{3}{2}\mu_k gt$$

When the ball stops sliding, 当球停止滑动时， $R\dot{\theta} = -\dot{y} \quad \Rightarrow \quad u_y = \mu_k gt \quad \Rightarrow \quad \dot{y} = 0$

$$y = u_y t - \frac{1}{2}\mu_k gt^2 = \frac{u_y^2}{2\mu_k g}$$

Hence the final velocity of the ball is 0, and its displacement is 因此球的最终速度为 0，其位移为 $y = \dfrac{u_y^2}{2\mu_k g}$.

3. Balloon (10 points) 气球（10 分）

The work done in stretching a spring is converted to its spring energy. Likewise, the work done in stretching a surface of a membrane is converted to its surface energy, given by $E = \gamma S$, where γ is called the *surface tension* of the membrane, and S is its surface area.
拉伸弹簧所做的功被转换成弹簧的内能。同样，拉伸一个薄膜表面所做的功被转换成它的表面能 $E = \gamma S$，其中 γ 称为薄膜的*表面张力*，而 S 是其表面面积。

(a) Consider a balloon of radius R. What is the change in surface energy when the radius changes by dR? Hence derive an expression for the pressure due to surface tension. (2 points)
考虑半径为R的气球。当半径改变为dR时，表面能的变化是多少？由此推导表面张力形成的压力的表达式。（2分）

The surface energy of the balloon is $E = \gamma 8\pi R^2$ (the balloon has both inner and outer surfaces).气球的表面能是$E = \gamma 8\pi R^2$（气球有里外两面）。

Hence $dE = \gamma 16\pi R dR$. Equating this to the work done by pressure 把这等同压强做的功

$$dW = pdV = p4\pi R^2 dR, \ \gamma 16\pi R dR = p4\pi R^2 dR \quad \Rightarrow \quad p = \frac{\gamma 16\pi R dR}{4\pi R^2 dR} = \frac{4\gamma}{R}.$$

(b) The surface tension of balloon A is γ. When it is filled with a diatomic ideal gas, its radius becomes R_0. The surface tension of balloon B is 2γ. When it is filled with the same kind of ideal diatomic gas, its radius becomes R_0. The temperature of the environment is T. The two balloons are then connected so that the gases are free to exchange between them until a steady state is reached. The final temperature is the same as that of the environment. What are the final radii of the two balloons respectively? You may neglect the atmospheric pressure in the analysis. (4 points)
气球A的表面张力为γ。当它充满了一种双原子的理想气体，其半径是R_0。气球B的表面张力为2γ。当它被相同的双原子理想气体充满时，其半径是R_0。环境的温度为T。然后两个气球被连接，使得气体可以在它们之间自由交流，直至达到稳定状态。最终温度与环境相同。问两个气球最终的半径分别是什么？在分析中你可以忽略大气压力。（4分）

Since the initial pressure in balloon B is higher, the gas will flow from balloon B to A. The radius of balloon B decreases and that of balloon A increases. Hence the pressure in balloon A and B increases and decreases respectively. The pressure difference increases, driving the system further away from equilibrium. This continues until all gases flow into balloon A. Hence $R_B = 0$.
因为气球B的初始压强较高，引致气体从气球B流向A。气球B的半径减少，气球A的半径增加。因此，气球A和B的压强分别增大和减小，使系统进一步远离平衡。这情况持续，直到所有的气体流入气球A。因此，$R_B = 0$。
To find R_A, we consider the initial number of moles of gas in balloon A:
要找出R_A，我们考虑起初时气球A中气体的摩尔数

$$n_A = \frac{p_A V_A}{RT} = \frac{1}{RT}\left(\frac{4\gamma}{R_0}\right)\left(\frac{4\pi R_0^3}{3}\right) = \frac{16\pi\gamma R_0^2}{3RT}.$$

Similarly, the initial number of moles of gas in balloon B:
同样，起初时气球B中气体的摩尔数：

$$n_B = \frac{p_B V_B}{RT} = \frac{1}{RT}\left(\frac{8\gamma}{R_0}\right)\left(\frac{4\pi R_0^3}{3}\right) = \frac{32\pi\gamma R_0^2}{3RT}.$$

Since the number of moles of gas is conserved, 因为气体的摩尔数守恒，

$$\frac{16\pi\gamma R_A^2}{3RT} = \frac{16\pi\gamma R_0^2}{3RT} + \frac{32\pi\gamma R_0^2}{3RT} \quad \Rightarrow \quad R_A = \sqrt{3}R_0.$$

(c) What are the amounts of heat gain by the gases in balloons A and B respectively during the gas exchange process in (b)? (4 points)
在(b)部的气体交流过程中，气球A和B增加的热能分别是什么？（4分）

Using the first law of thermodynamics, 应用热力学第一定律，
Heat gain = internal energy change + work done by the gas
热能增加 = 内能改变 + 气体做的功
The internal energy of an ideal gas is independent of its volume.
理想气体的内能与体积无关。
For balloon A, 对气球A来说，

Internal energy change: 内能改变： $\Delta U_A = \frac{16\pi\gamma}{3RT}\left(\sqrt{3}R_0\right)^2 c_V T - \frac{16\pi\gamma R_0^2}{3RT} c_V T = \frac{32\pi c_V \gamma R_0^2}{3R}$.

Work done by the gas is equal to the change in surface energy of the balloon:
气体做的功等于气球表面能的改变： $W_A = \gamma 8\pi\left(\sqrt{3}R_0\right)^2 - \gamma 8\pi R_0^2 = 16\pi\gamma R_0^2$.

For diatomic ideal gases, 在双原子的理想气体中， $c_V = \frac{5}{2}R$.

Hence heat gain: 所以热能增加是： $Q_A = \Delta U_A + W_A = \frac{32\pi\gamma R_0^2}{3R}\left(\frac{5}{2}R\right) + 16\pi\gamma R_0^2 = \frac{128\pi\gamma R_0^2}{3}$.

Similarly, for balloon B, 同样，对气球B来说，

$\Delta U_B = -\frac{32\pi\gamma R_0^2}{3RT} c_V T = -\frac{32\pi c_V \gamma R_0^2}{3R}$.

$W_B = -\gamma 16\pi R_0^2$.

$Q_B = \Delta U_B + W_B = -\frac{32\pi\gamma R_0^2}{3R}\left(\frac{5}{2}R\right) - 16\pi\gamma R_0^2 = -\frac{128\pi\gamma R_0^2}{3}$.

Remark: A common mistake is to assume that the pressures in both balloons are the same when the gas exchange process has reached steady state. This implies
注：一个常见的错误，是假设当气体交流过程达到稳定状态时，两气球的压强相同。这显示

$\frac{4\gamma}{R_A} = \frac{8\gamma}{R_B} \Rightarrow R_B = 2R_A$.

Since the number of moles of gas is conserved, 因为气体的摩尔数守恒，

$R_A^2 + 2(2R_A)^2 = 3R_0^2 \Rightarrow R_A = \frac{R_0}{\sqrt{3}}$ and $R_B = \frac{2R_0}{\sqrt{3}}$ and $Q_A = -\frac{128\pi\gamma R_0^2}{9}$, $Q_B = \frac{128\pi\gamma R_0^2}{9}$.

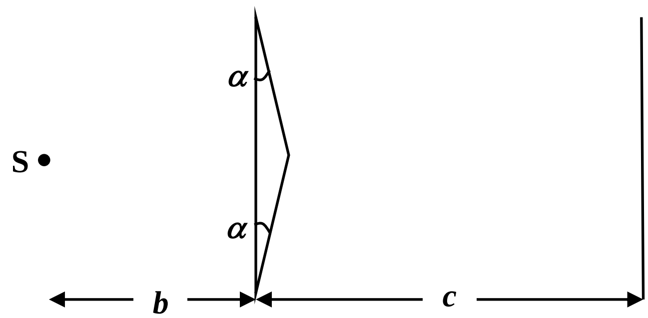
However, this equilibrium state is unstable. 可是，这平衡态是不稳定的。

4. Fresnel Biprism (10 points) 菲涅耳双棱镜（10 分）

Fresnel biprism was devised shortly after the famous Young's double slit experiment to confirm the interference phenomenon. Nowadays, it is widely used in different applications. As shown in the figure, it consists of a single light source S and a pair of wedge-shaped prisms arranged back to back. We introduce the following notations:

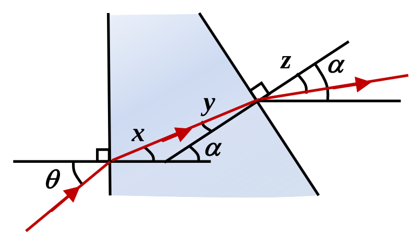
在著名的杨氏双缝实验面世后不久，便产生了菲涅耳双棱镜的设计，用以确认干涉现象。如今，它被广泛用于不同的应用。如图所示，它由一个单一的光源 S 和一对背对背的楔形棱镜组成。我们引入以下符号：

n = refractive index of the biprism 双棱镜的折射率
α = apex angle of each prism 双棱镜的顶角
b = distance between light source and biprism 光源与双棱镜的距离
c = distance between biprism and screen 双棱镜与屏幕的距离
λ = wavelength of light 光的波长



(a) Derive an expression for the angular deviation after a light beam has passed through one of the two prisms. (3 points)
试推导光束经过其中一个棱镜后偏转角的表达式。（3 分）

Consider a light beam incident on the upper prism. Let θ be the incident angle. For the angles shown in the figure,
考虑入射上棱镜的光束。设θ为入射角。对图中显示的角度来说，



$$x = \frac{\theta}{n}, \quad y = \alpha - x = \alpha - \frac{\theta}{n}, \quad z = ny = n\alpha - \theta.$$

The angle between the deflected beam and the horizontal direction 偏转光束与平行方向的角
$= \alpha - z = \theta - (n-1)\alpha$.

Hence the angular deviation of the beam is 所以光束的偏转角是$(n – 1)\alpha$.

(b) Derive an expression for the separation of the fringes on the screen. (4 points)
试推导屏幕上条纹距离的表达式。（4 分）

When $\theta = (n – 1)\alpha$, the beam emerges parallel to the horizontal direction. Hence the image of S in the upper prism is located at a distance $b\theta = (n – 1)b\alpha$ above S.
当$\theta = (n – 1)\alpha$，光束在偏转后在以水平方向出现。因此 S 在上棱镜中的影像位于 S 之上，距离为 $b\theta = (n – 1)b\alpha$。
Similarly, the image of S in the lower prism is located at $(n – 1)b\alpha$ below S.
同样地，S 在下棱镜中的影像位于 S 之下，距离为 $b\theta = (n – 1)b\alpha$。
This is equivalent to the Young's double slit experiment with slit separation $d = 2(n – 1)b\alpha$.
这布局等同杨氏双缝实验，双缝距离为 $d = 2(n – 1)b\alpha$。
Condition for constructive interference: 相长干涉的条件： $d\sin\theta = m\lambda$.

Positions of the bright fringes:光条纹的位置： $y=(b+c)\tan\theta\approx(b+c)\sin\theta=(b+c)\dfrac{m\lambda}{d}$.

Hence the fringe separation:因此条纹间的距离： $\Delta y=(b+c)\dfrac{\lambda}{d}=\dfrac{(b+c)\lambda}{2(n-1)b\alpha}$.

(c) In a modern application on electron microscopes, the single light source is replaced by a parallel beam of wave incident normally to the flat surface of the biprism. Derive an expression for the separation of the fringes on the screen. (3 points)
在现代，这原理已应用到电子显微镜中。在这应用中，单个光源被替换成入射的平行波束，垂直于双棱镜的平面。试推导屏幕上条纹距离的表达式。（3 分）

When the waves are incident on the screen at an angle ϕ, it sets up a traveling wave in the transverse direction with a wavelength $\lambda/\sin\phi$.
当波以角度ϕ入射到屏幕上，它产生了一个波长为$\lambda/\sin\phi$的横向行波。
In the biprism setup, we have $\phi=(n-1)\alpha$ and two waves with the same frequency and wavelength traveling in opposite directions. Hence a standing wave is formed.
在双棱镜设置中，我们有$\phi=(n-1)\alpha$，和两个具有相同频率和波长、但沿相反方向行进的行波。因此，驻波便形成了。
Fringes in standing waves are separated by half wavelengths.
驻波中条纹的距离为波长的一半。
Hence the fringe separation is: 因此，条纹间距为: $\Delta y=\dfrac{\lambda}{2\sin\phi}\approx\dfrac{\lambda}{2(n-1)\alpha}$.

5. Ionic Crystals (11 points) 离子晶体（11 分）

An ionic crystal can be modeled by a chain of positively and negatively charged ions. The ionic separation is a. The positive ions with atomic mass M are located at the positions $x = na$ where n is even. The negative ions with atomic mass m ($m < M$) are located at the positions $x = na$ where n is odd. The ions are coupled to their neighbors by springs, which provide restoring forces to their transverse displacements. The returning force is proportional to the displacements of the ions relative to their neighbors, and the spring constant is k.
我们可以一串带正电和带负电的离子，作为离子晶体的模型。离子间的距离为 a。正离子的原子质量为 M，处于位置 $x = na$，其中 n 是偶数。负离子的原子质量为 m（$m < M$），处于在位置 $x = na$，其中 n 是奇数。相邻的离子有弹簧耦合，弹簧为离子的横向位移提供返回力。返回力正比于离子相对于相邻离子的位移，并且弹簧常数为 k。

(a) Let $u_n(t)$ be the transverse displacement of the ion at $x = na$ and time t. Derive the equations of motion for both types of ions. Show that the solution of the equation of motion can be written as
令 $u_n(t)$为处于 $x= na$ 的离子在时间 t 的横向位移。试推导两种类型离子的运动方程。表明运动方程的解可以写成

$$u_n(t)=\begin{cases}A_M\sin(qna-\omega t) & n\text{ even,}\\ A_m\sin(qna-\omega t) & n\text{ odd.}\end{cases}$$

Find the relation between q and ω. (3 points) 试找出 q 与ω的关系。（3 分）

Using Newton’s law,利用牛顿定律，

$M\ddot{u}_n = k(u_{n+1}-u_n)-k(u_n-u_{n-1}) = ku_{n-1}-2ku_n+ku_{n+1}$ for n even, n 是偶数。

$m\ddot{u}_n = k(u_{n+1}-u_n)-k(u_n-u_{n-1}) = ku_{n-1}-2ku_n+ku_{n+1}$ for n odd. n 是奇数。

For even n, n 是偶数时，

$$-M\omega^2 A_M\sin(qna-\omega t)$$
$$= kA_m\sin[q(n-1)a-\omega t]-2kA_M\sin(qna-\omega t)+kA_m\sin[q(n+1)a-\omega t]$$
$$= 2kA_m\cos qa\sin(qna-\omega t)-2kA_M\sin(qna-\omega t)$$
$$\Rightarrow\quad -M\omega^2 A_M = 2kA_m\cos qa-2kA_M$$
$$\Rightarrow\quad (M\omega^2-2k)A_M+2k\cos qaA_m = 0. \qquad (1)$$

For odd n, n 是奇数时，

$$-m\omega^2 A_m\sin(qna-\omega t)$$
$$= kA_M\sin[q(n-1)a-\omega t]-2kA_m\sin(qna-\omega t)+kA_M\sin[q(n+1)a-\omega t]$$
$$= 2kA_M\cos qa\sin(qna-\omega t)-2kA_m\sin(qna-\omega t)$$
$$\Rightarrow\quad -m\omega^2 A_m = 2kA_M\cos qa-2kA_m$$
$$\Rightarrow\quad (m\omega^2-2k)A_m+2k\cos qaA_M = 0. \qquad (2)$$

Eqs. (1) and (2) have non-trivial solutions if 方程（1）和（2）有非零解的条件是

$$\frac{A_M}{A_m} = -\frac{2k\cos qa}{M\omega^2-2k} = -\frac{m\omega^2-2k}{2k\cos qa}$$
$$\Rightarrow\quad (M\omega^2-2k)(m\omega^2-2k) = 4k^2\cos^2 qa$$
$$\Rightarrow\quad Mm\omega^4-2k(M+m)\omega^2+4k^2\sin^2 qa) = 0$$
$$\Rightarrow\quad \omega = \sqrt{\frac{k(M+m)\pm k\sqrt{(M+m)^2-4Mm\sin^2 qa}}{Mm}}.$$

(b) Find the solutions of ω in the limit q = 0, and the relation between A_M and A_m for each solution. (2 points)

在极限 $q = 0$，求ω的所有解，并且求在每个解中 A_M 与 A_m 间的关系。（2 分）

In the limit $q = 0$, the high frequency solution is 在极限 $q = 0$，高频解是

$$\omega^2 = \frac{k(M+m)+k\,|\,M+m\,|}{Mm} = \frac{2k(M+m)}{Mm} \quad\Rightarrow\quad \omega = \sqrt{\frac{2k(M+m)}{Mm}}.$$
$$\frac{A_M}{A_m} = -\frac{2k\cos qa}{M\omega^2-2k} = -\frac{2k}{2k(M+m)/m-2k} = -\frac{m}{M}.$$

The motions of the two ions are out of phase. 两种离子的运动是反相的。

The low frequency solution is 低频解是 $\omega^2 = \dfrac{1}{Mm}\left[k(M+m)-k(M+m)\sqrt{1-\dfrac{4Mm(qa)^2}{(M+m)^2}}\right]$

$$\approx \frac{k(M+m)}{Mm}\left[1-\left(1-\frac{2Mm(qa)^2}{(M+m)^2}\right)\right] = \frac{2kq^2a^2}{M+m} \quad\Rightarrow\quad \omega = \sqrt{\frac{2k}{M+m}}qa.$$
$$\frac{A_M}{A_m} = -\frac{2k\cos qa}{M\omega^2-2k} = -\frac{2k}{-2k} = 1.$$

The motions of the two ions are in phase. 两种离子的运动是同相的。

(c) In the limit $q = 0$, calculate the wave velocity of the low frequency mode. (1 point)
在极限 q =0，试计算低频模式的波速。（1 分）

Wave velocity: 波速：$v = \dfrac{\omega}{q} = a\sqrt{\dfrac{2k}{M+m}}$.

(d) In the limit $q = \pi/2a$, find the solutions of ω, and the relation between A_M and A_m for each solution. (2 points)
在极限 $q = \pi/2a$，求ω的所有解，并且求在每个解中 A_M 与 A_m 间的关系。（2 分）

In the limit $q = \pi/2a$, the high frequency solution is 在极限 $q = \pi/2a$，高频解是

$$\omega^2 = \frac{k(M+m)+k|M-m|}{Mm} = \frac{2k}{m} \quad \Rightarrow \quad \omega = \sqrt{\frac{2k}{m}}.$$

$$\frac{A_M}{A_m} = -\frac{2k\cos qa}{M\omega^2 - 2k} = 0.$$

Only the lighter ion moves.只有较轻的离子在运动。

The low frequency solution is 低频解是 $\omega^2 = \dfrac{k(M+m)-k|M-m|}{Mm} = \dfrac{2k}{M} \quad \Rightarrow \quad \omega = \sqrt{\dfrac{2k}{M}}$.

$$\frac{A_M}{A_m} = -\frac{2k\cos qa}{M\omega^2 - 2k}.$$

Since both the denominator and numerator vanish, we have to consider higher order terms.
因为分母和分子同时消失，我们必须考虑高阶项

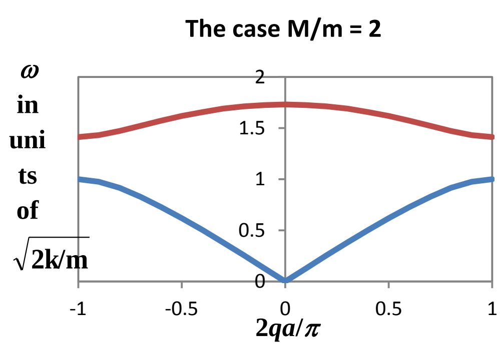
$$\omega^2 = \frac{k(M+m)-k\sqrt{(M-m)^2+4Mm\cos^2 qa}}{Mm} = \frac{k}{Mm}\left[M+m-(M-m)\sqrt{1+\frac{4Mm\cos^2 qa}{(M-m)^2}}\right]$$

$$\approx \frac{k}{Mm}\left[M+m-(M-m)-\frac{2Mm\cos^2 qa}{M-m}\right] = \frac{2k}{M}\left[1-\frac{M\cos^2 qa}{M-m}\right]$$

$$\Rightarrow \quad \frac{A_M}{A_m} \approx -\frac{2k\cos qa}{2k-2kM\cos^2 qa/(M-m)-2k} = \frac{M-m}{M\cos qa} \to \infty.$$

Only the more massive ion moves. 只有较重的离子在运动。

(e) Sketch the angular frequency ω as a function of the wavenumber q from $q = -\pi/2a$ to $q = \pi/2a$. (2 points)
试绘出角频率ω作为波数 q 的函数的草图，范围从 $q = -\pi/2a$ 到 $q = \pi/2a$。（2 分）



(f) An electromagnetic wave is incident on the crystal. Which frequency mode will be excited? (1 point) 有电磁波入射到晶体。哪种频率模式会被激发？（1 分）

Since in the high frequency mode, positive and negative ions oscillate out of phase, oscillating electric dipole moments will be formed. Hence the high frequency mode will be excited. 由于在高频率的模式中，正负离子以反相振动，振动电偶极矩将形成。因此，高频模式将被激发。

《THE END 完》

Pan Pearl River Delta Physics Olympiad 2015
2015 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
香港科技大学高等研究院赞助
Part-2 (Total 2 Problems) 卷-2（共2 题）
(2:00 pm – 5:00 pm, 25 February, 2015)

1. Exoplanet Microlensing (25 points) 系外行星的微透镜效应（25 分）

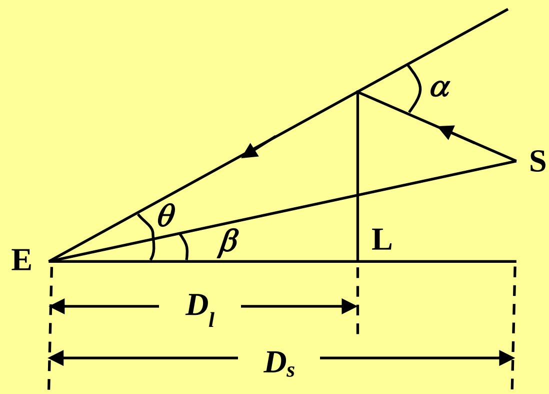
Reference: 参考文献：B. S. Gaudi, Exoplanet Microlensing, EXOPLANETS, edited by S. Seager, Space Science Series of the University of Arizona Press (Tucson, AZ, 2010).

With the discovery of planets orbiting around stars in recent years, the observation of exoplanets from astronomical distances became a challenge to scientists. Gravitational microlensing is one of the detection methods. It makes use of Einstein's discovery in general relativity that when a light ray passing near a spherically symmetric body of mass M, its direction will be deflected towards the body by a small angle given by
随着近年发现不少绕着恒星运行的行星，怎样观察相隔天文距离的系外行星便成为科学家的挑战。引力微透镜是其中一种检测方法。它利用爱因斯坦在广义相对论里发现的原理，就是当光线经过一个质量为 M 的球对称物体时，方向会朝向物体偏转，偏转的小角度为

$$\alpha = \frac{4GM}{rc^2},$$

where G is the gravitational constant, c is the speed of light, and r is the distance of closest approach of the light ray to the body. In this problem, we will study the principle of detecting exoplanet by microlensing.
其中 G 是万有引力常数，c 是光速，r 是光线和物体的最短距离。在这个问题中，我们将研究通过微透镜效应探测系外行星的原理。

(a) Consider a distant star S located at a distance D_s from Earth E, acting as the light source. Another star L of mass M and located at distance D_l ($< D_s$) from Earth acts as the lens. The lines EL and ES make a small angle β between them. Construct the following sketch in the answer book: (a1) the line EL, (a2) the line ES, (a3) the distances D_l and D_s, (a4) the angle β (remark: although this angle is small in practice, it should not be drawn too small for the purpose of clarity), (a5) a line perpendicular to EL through L, acting as the gravitational lens, (a6) the light ray from S to E, assuming that each of the segments between S and the lens and that between the lens and E are straight lines, (a7) the deflection angle α, (a8) the apparent angle θ of the star S as observed on Earth (relative to line EL). (3 marks)
考虑一个遥远的恒星 S，离地球 E 的距离为 D_s，作为光源。另一颗恒星 L，质量为 M，离地球的距离为 D_l（$< D_s$），作为透镜。线 EL 和 ES 间的小角度为β。试在答题簿上绘出以下草图：（a1）线 EL，（a2）线 ES，（a3）距离 D_l 和 D_s，（a4）角度β（注：虽然该角度实际上很小，但为清楚起见，不应把它绘得太小），（a5）一条垂直于 EL 而通过 L 的线，作为引力透镜，（a6）从 S 到 E 的光线，假定 S 和透镜之间的线段及透镜和 E 之间的线段各可视作直线，（a7）偏转角α，（a8）从地球观察星 S 的视角θ（相对于线 EL）。（3 分）



(b) Derive an equation for the angle θ in terms of the parameters D_s, D_l, G, M, c and β, assuming that all angles are small. (3 points)
试推导θ的方程式，以参数 D_s, D_l, G, M, c 和 β表达，可假设所有角度都很小。（3 分）

Constructing the vertical line XY through S, we have
作一垂直线 XY 通过 S，我们得
XY = XS + SY,

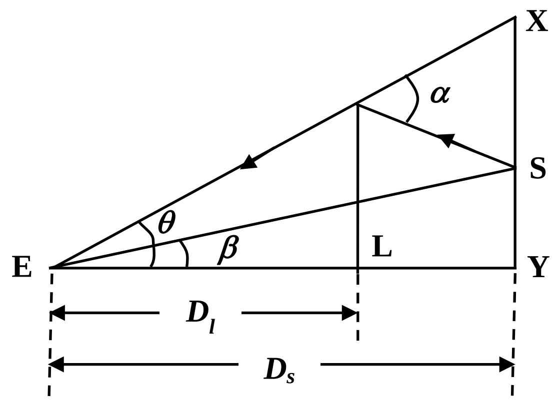
$$D_s\theta = D_s\beta + (D_s - D_l)\alpha$$

Substituting $\alpha = \frac{4GM}{rc^2}$ and $r = D_l\theta$,

代入$\alpha = \frac{4GM}{rc^2}$ 和 $r = D_l\theta$，

$$\theta = \beta + \left(\frac{D_s - D_l}{D_s D_l}\right)\frac{4GM}{c^2\theta}.$$

$$\beta = \theta - \frac{4GM}{c^2\theta}\left(\frac{D_s - D_l}{D_s D_l}\right).$$



(c) Consider the case that the lens is exactly aligned with the source ($\beta = 0$). The image of S appears to be a ring known as an Einstein ring. Derive the expression for the angular radius θ_E of the Einstein ring. (2 points)
考虑透镜与光源对準的情况（$\beta = 0$）。 S 的影象呈环形，称为爱因斯坦环。试推导爱因斯坦环的角半径θ_E 的表达式。 （2 分）

$$0 = \theta - \frac{4GM}{c^2\theta}\left(\frac{D_s - D_l}{D_s D_l}\right) \Rightarrow \theta_E = \sqrt{\frac{4GM}{c^2}\left(\frac{D_s - D_l}{D_s D_l}\right)}.$$

(d) Calculate the Einstein radius for the following typical values:
试以下列的典型值，计算爱因斯坦半径：
M = 0.3 solar mass, D_s = 10 kpc. D_l = 3 kpc.
Give your answer in milli-arc-seconds. You may use the following constants:
请以 milli-arc-seconds 表达你的答案。您可以使用以下参量：
$G = 6.67 \times 10^{-11}$ Nm^{2}kg^{-2}, 1 solar mass = 1.99×10^{30} kg, $c = 3 \times 10^8$ ms^{-1}, 1 kpc = 3.09×10^{19} m, 1 radian = 206265 arc seconds. (1 point) （1 分）

$$\theta_E = \sqrt{\frac{4(6.67\times10^{-11})(0.3)(1.99\times10^{30})}{(3\times10^8)^2}\left[\frac{10-3}{(10)(3)(3.09\times10^{19})}\right]} = 3.66\times10^{-9}\text{ rad}$$

= 0.754 milli-arc-seconds.

(e) When the lens and the source are not exactly aligned, there will be two images of S. It is convenient to express the angles β and θ in multiples of the Einstein radius θ_E. Hence we define $u\equiv\frac{\beta}{\theta_E}$ and $y\equiv\frac{\theta}{\theta_E}$. Derive the expressions for the angular positions y of the two images in terms of u. (2 points)

当透镜和光源不完全对齐时，S 将有两个影像。为方便起见，我们以爱因斯坦半径θ_E的倍数表达角β 和θ。因此我们定义$u\equiv\frac{\beta}{\theta_E}$和$y\equiv\frac{\theta}{\theta_E}$。试推导两个影像的角位置 y，以 u 表示。（2 分）

$$\beta=\theta-\frac{4GM}{c^2\theta}\left(\frac{D_s-D_l}{D_sD_l}\right) \Rightarrow \beta=\theta-\frac{\theta_E^2}{\theta} \Rightarrow u=y-\frac{1}{y} \Rightarrow y^2-uy-1=0 \Rightarrow$$

$$y=\frac{u\pm\sqrt{u^2+4}}{2}.$$

(f) To study the effect of the finite size of star S, we introduce Cartesian coordinates on the plane normal to ES and through S, with the y axis lying in the plane containing E, L and S. Consider the corners $(0, u+\delta)$ and (δ, u) of a square on the surface of star S ($\delta << u$). Calculate the coordinates of the two corners of the two images when viewed from Earth. (2 points)

为研究星 S 有限大小的影响，我们在垂直于 ES 和通过 S 的平面上，引入一平面直角坐标，其中 y 轴位于包含 E，L 和 S 的平面中。考虑星 S 表面上一个正方形的角$(0, u+\delta)$ 和 (δ, u) ($\delta << u$)。试计算从地球观察时，这两个影像的两个角的坐标。（2 分）

The image of (δ, u) is $\left(\frac{y}{u}\delta, y\right)$. (δ, u)的影像是$\left(\frac{y}{u}\delta, y\right)$。

$$y=\frac{u\pm\sqrt{u^2+4}}{2} \Rightarrow \delta y=\frac{1}{2}\left(1\pm\frac{u}{\sqrt{u^2+4}}\right)\delta u=\frac{\sqrt{u^2+4}\pm u}{2\sqrt{u^2+4}}\delta u=\pm\frac{y\delta u}{\sqrt{u^2+4}}.$$

Hence the image of $(0, u+\delta)$ is 所以$(0, u+\delta)$的影像是$\left(0, \frac{u\pm\sqrt{u^2+4}}{2}\left[1\pm\frac{\delta}{\sqrt{u^2+4}}\right]\right)$.

In summary, for the image at $y=\frac{u+\sqrt{u^2+4}}{2}$, the images of the corners $(0, u+\delta)$ and (δ, u) are respectively

总结一下，对于在 $y=\frac{u+\sqrt{u^2+4}}{2}$ 的影像，角$(0, u+\delta)$和(δ, u)的影像分别是

$$\left(0, \frac{u+\sqrt{u^2+4}}{2}\left[1+\frac{\delta}{\sqrt{u^2+4}}\right]\right) \text{ and } \left(\frac{u+\sqrt{u^2+4}}{2u}\delta, \frac{u+\sqrt{u^2+4}}{2}\right).$$

For the image at $y=\dfrac{u-\sqrt{u^2+4}}{2}$, the images of the corners (0, u + δ) and (δ, u) are respectively 对于在 $y=\dfrac{u-\sqrt{u^2+4}}{2}$, 的影像， 角(0, u + δ)和(δ, u)的影像分别是

$$\left(0,\frac{u-\sqrt{u^2+4}}{2}\left[1-\frac{\delta}{\sqrt{u^2+4}}\right]\right) 和 \left(\frac{u-\sqrt{u^2+4}}{2u}\delta,\frac{u-\sqrt{u^2+4}}{2}\right).$$

(g) Calculate the areal magnifications of the two images of star S in terms of u. Following the practice in astronomical observations, give your answer in absolute values. (2 points)
试计算星 S 的两个影像的面积放大率，请以 u 表达。按照天文观测的习惯，请以绝对值为答案。（2 分）

The length scales are magnified by $\dfrac{u\pm\sqrt{u^2+4}}{2u}$ and $\dfrac{u\pm\sqrt{u^2+4}}{\pm 2\sqrt{u^2+4}}$ in the x and y directions respectively. Hence the areal magnifications are:
长度分别在 x 和 y 方向放大了 by $\dfrac{u\pm\sqrt{u^2+4}}{2u}$ 和 $\dfrac{u\pm\sqrt{u^2+4}}{\pm 2\sqrt{u^2+4}}$。因此，面积放大率为:

$$\left|\left(\frac{u\pm\sqrt{u^2+4}}{2u}\right)\left(\frac{u\pm\sqrt{u^2+4}}{\pm 2\sqrt{u^2+4}}\right)\right|=\frac{\left(u\pm\sqrt{u^2+4}\right)^2}{4|u|\sqrt{u^2+4}}=\frac{1}{2}\left(\frac{u^2+2}{|u|\sqrt{u^2+4}}\pm 1\right).$$

(h) In practice, since the images cannot be resolved, astronomers measure the sum of the magnifications of the two images. Derive the expression for the total magnification. Describe its behavior when star S is remote (u approaches infinity) and when S approaches perfect alignment with L and E (u approaches 0). (3 points)
实际上，由于影象不易分辨，天文学家只测量两个影像的放大率的总和。试推导总放大率的表达式。试描述星 S 在远处时（u 趋近无穷大），及星 S 趋近对準 L 与 E 时(u 趋近 0)，总放大率的行为。（3 分）

Total magnification: 总放大率:

$$A=\frac{1}{2}\left(\frac{u^2+2}{|u|\sqrt{u^2+4}}+1\right)+\frac{1}{2}\left(\frac{u^2+2}{|u|\sqrt{u^2+4}}-1\right)=\frac{u^2+2}{|u|\sqrt{u^2+4}}.$$

When u approaches infinity, 当 u 趋近无穷大： $A\to 1$.

When u approaches 0, 当 u 趋近 0： $A\to\dfrac{1}{|u|}$.

(i) A planet P of star L has mass m and is located in the plane of E, L and S at the same distance D_l from Earth. EP and EL makes an angle θ_p. Derive an equation for the angle θ taking into account the gravitational lensing effects of both star L and planet P. Expressions in the equation should be written in terms of the parameters D_s, D_l, G, M, c, β, m and θ_p, assuming that all angles are small. Simplify the equation by introducing the mass ratio $q\equiv\dfrac{m}{M}$ and the rescaled positions $u_p\equiv\dfrac{\theta_p}{\theta_E}$, $u\equiv\dfrac{\beta}{\theta_E}$, $y\equiv\dfrac{\theta}{\theta_E}$. (3 points)

星 L 旁有一行星 P 位于 E、L 和 S 的平面上，其质量为 m，与地球距离跟星 L 同为 D_l，EP 与 EL 间角度为θ_p 。考虑到星 L 和行星 P 两者的引力透镜作用，试推导角θ的方程式，式中的表达式应以 D_s、D_l、G、 M、 c、β、 m 和θ_p 表达。可假设所有角度都很小。引入质量比 $q \equiv \frac{m}{M}$ 和重整位置 $u_p \equiv \frac{\theta_p}{\theta_E}$，$u \equiv \frac{\beta}{\theta_E}$，$y \equiv \frac{\theta}{\theta_E}$， 以简化方程式。（3 分）

As shown in the figure,如图所示，

XZ = XY + YS + SZ.

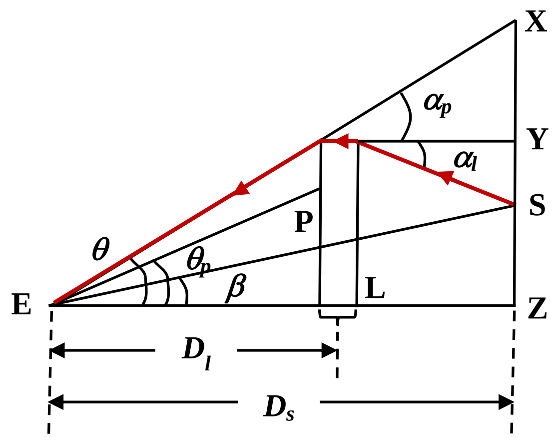
$D_s\theta = (D_s - D_l)\alpha_p + (D_s - D_l)\alpha_l + D_s\beta.$

Substituting 代入 $\alpha_l = \frac{4GM}{r_l c^2}$,

$\alpha_p = \frac{4Gm}{r_p c^2}$, $r_l = D_l\theta$, $r_p = D_l(\theta - \theta_p)$,

$$\theta = \beta + \left(\frac{D_s - D_l}{D_s D_l}\right)\frac{4G}{c^2}\left(\frac{M}{\theta} + \frac{m}{\theta - \theta_p}\right).$$

$$\beta = \theta - \frac{4G}{c^2}\left(\frac{D_s - D_l}{D_s D_l}\right)\left(\frac{M}{\theta} + \frac{m}{\theta - \theta_p}\right).$$



The equation can be simplified to 方程式可简化为

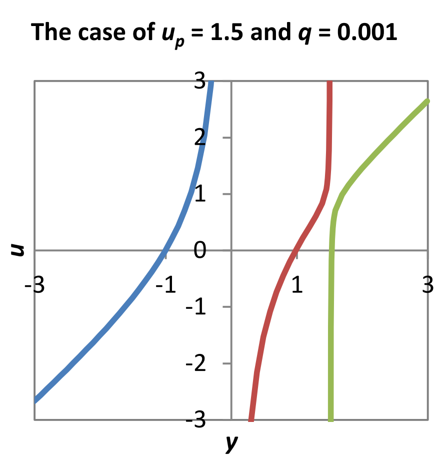
$$u = y - \frac{1}{y} - \frac{q}{y - u_p}.$$

(j) In typical exoplanet detections, there is a motion of star S relative to star L. As star S approaches the closest distance to star L and moves away, u decreases with time to a minimum value u_0 and increases again. By plotting the magnification of the image of star S versus time, one observes a smooth and relatively broad peak in the magnification curve due to gravitational lensing by star L. In addition, one can observe a side peak due to the presence of the planet. For $q << 1$, estimate the width of this side peak, that is, the range of u in which the side peak is significant. (1 point)

在典型的系外行星检测中，星 S 对于星 L 有相对运动。星 S 趋近星 L 至最短距离，然后离开，过程中 u 随时间降到最小值 u_0 然后再增加。把星 S 影像的放大率与时间的关系绘成图表，放大率曲线上可以看到一个平滑和较宽的主峰，是由星 L 的引力透镜作用形成的。另外，我们可以观察到一个侧峰，是由行星形成的。对于 $q << 1$，试估计这个侧峰的宽度，也就是可以显著看到侧峰的 u 数值范围。 （1 分）

Plotting u as a function of y, we see in the figure shown that there are three solutions for each given u. The side peak is significant when the light rays fall within the Einstein radius of the planet. Following part (c), the Einstein radius is proportional to $\sqrt{M}$. Hence the width of the side peak is of the order $\sqrt{q}$.

把 u 作为 y 的函数绘成图表，我们看到对每个给定的 u 有三个解。当光线落在行星的爱因斯坦



半径范围内，侧峰便变为显著。从(c)部得知，爱因斯坦半径与$\sqrt{M}$成正比。因此，侧峰的宽度的量级为$\sqrt{q}$。

(k) For $q << 1$, consider the situation that light rays pass very near to planet P, so that the gravitational lensing by star L becomes relatively insignificant. Calculate the position of star S where the total magnification of its image diverges, and the behavior of the total magnification in the neighborhood of this location. (3 points)
当 $q << 1$ 时，考虑光线非常靠近行星 P 的情况，在这情况下星 L 的引力透镜作用相对很弱。试计算当星 S 图像的总放大率发散时星 S 的位置，和这位置附近总放大率的行为。（3 分）

When light rays pass very near to planet P, $y \approx u_p$. The equation for y becomes
当光线非常靠近行星 P 时，$y \approx u_p$。y 的方程式变成

$$u \approx y - \frac{q}{y - u_p}.$$

Let $y' = y - u_p$ and $u' = u - u_p$. Then the equation becomes $u' \approx y' - \frac{q}{y'}$.

设 $y' = y - u_p$ 和 $u' = u - u_p$。则方程式变成 $u' \approx y' - \frac{q}{y'}$.

The solutions are 方程式的解是 $y' \approx \frac{u' \pm \sqrt{u'^2 + 4q}}{2}$.

When $u' = 0$, $y' \approx \pm\sqrt{q}$, confirming that the Einstein radius is $\sqrt{q}$.
$u' = 0$ 时，$y' \approx \pm\sqrt{q}$，确认爱因斯坦半径为 $\sqrt{q}$.

Following parts (f) and (g), the areal magnifications are 跟随(f)和(g)部，面积放大率为

$$\left|\frac{y'}{u'}\left(\frac{dy'}{du'}\right)\right| \approx \left|\frac{u' \pm \sqrt{u'^2+4q}}{2u'}\frac{1}{2}\left(1 \pm \frac{u'}{\sqrt{u'^2+4q}}\right)\right| = \frac{\left(u' \pm \sqrt{u'^2+4q}\right)^2}{4|u'|\sqrt{u'^2+4q}} = \frac{1}{2}\left(\frac{u'^2+2q}{|u'|\sqrt{u'^2+4q}} \pm 1\right)$$

Total magnification: 总放大率：

$$A = \frac{1}{2}\left(\frac{u'^2+2q}{|u'|\sqrt{u'^2+4q}} + 1\right) + \frac{1}{2}\left(\frac{u'^2+2q}{|u'|\sqrt{u'^2+4q}} - 1\right) = \frac{u'^2+2q}{|u'|\sqrt{u'^2+4q}}.$$

Hence the total magnification diverges when star S is located at $u' = 0$, or $u = u_p$.
因此当 $u' = 0$ 或 $u = u_p$ 时，总放大率发散。

When u' approaches 0, 当 u' 趋近 0： $A \to \frac{\sqrt{q}}{|u'|} = \frac{\sqrt{q}}{|u - u_p|}$.

2. Cosmic Gravitational Waves (25 points) 宇宙引力波（25 分）

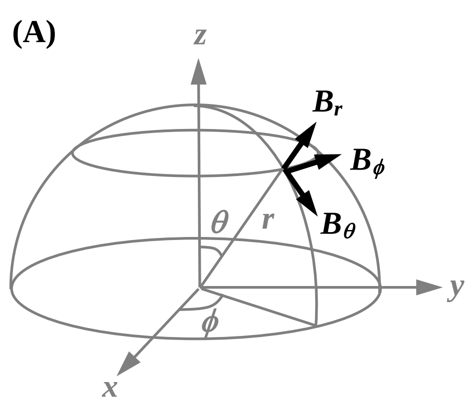
In March 2014, scientists operating gravitational wave detectors in the South Pole claimed that they found evidences of gravitational waves originated from the early universe in the cosmic microwave background radiation. While the evidence is still being debated, it is interesting to understand how gravitational waves interact with electromagnetic (EM) waves. To approach this issue, we start by considering how molecules scatter EM waves.

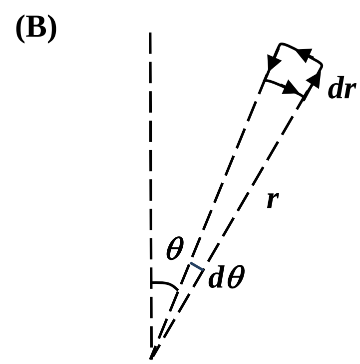
2014 年 3 月，操作南极引力波探测器的科学家，声称在宇宙微波背景辐射中，发现来自早期宇宙的引力波的证据。虽然证据还存在争议，但了解引力波如何作用于电磁（EM）波是一个有趣的课题。为了处理这个问题，我们首先考虑分子是如何散射电磁波。

(a) An oscillating electric dipole consists of charges oscillating at an angular frequency ω. Specifically, the charges are $Q(t) = \pm Q_0\cos\omega t$, located at $(x, y, z) = (0, 0, \pm s)$ respectively. What is the current between them? (1 point)
一个振动的电偶极子，包含以角频率ω振动的电荷。具体来说，电荷分别为 $Q(t) = \pm Q_0\cos\omega t$, 位于$(x, y, z) = (0, 0, \pm s)$。它们之间的电流是什么？（1 分）

$$I(t) = \frac{d}{dt}\left(Q_0 \cos \omega t\right) = -\omega Q_0 \sin \omega t.$$

(b) In spherical coordinates, we denote the components of the magnetic field as B_r, B_θ and B_ϕ, as shown in figure (A). Calculate $B_\phi(r, \theta, t)$ according to Biot-Savart's law at time t and distance r from the origin making an angle θ with the z axis. Note that due to the finite speed of light c, the magnetic field at a distant location is due to the time-changing current at an earlier instant. Hence the *retarded* magnetic field takes the form $\mathbf{B}(\mathbf{r},t) = \mathbf{B}_0(\mathbf{r})\cos(\omega t - kr + \psi)$, where $k \equiv \frac{\omega}{c}$ is the *wavenumber*, and ψ is the phase shift. Express your answer in terms of the magnitude of the dipole moment $p \equiv 2Qs$ in the limit s approaches 0. Below, your answer to this part will be denoted as $B_{BS}(r, \theta, t)$. (3 points)
在球坐标中，我们以 B_r, B_θ 和 B_ϕ,表示磁场的分量，如图（A）所示。根据毕奥 - 萨伐尔定律，试计算磁场 $B_\phi(r, \theta, t)$，其中 r 为位置与原点的距离，θ为位置与 z 轴形成的角，t 为时间。注意，由于光以有限速率 c 传播，在远处的磁场是源于某一较早时刻的电流（电流随时间变化）。因此，*延迟*磁场的形式为$\mathbf{B}(\mathbf{r},t) = \mathbf{B}_0(\mathbf{r})\cos(\omega t - kr + \psi)$，其中 $k \equiv \frac{\omega}{c}$ 是*波数*，而ψ是相移。答案请以偶极矩 $p \equiv 2Qs$ 表达（取 s 趋于 0 的极限）。下面，你在这部的答案将被表示为 $B_{BS}(r, \theta, t)$。（3 分）





Using Biot-Savart's law,利用根据毕奥 - 萨伐尔定律，

$$\mathbf{B}(r,\theta,t) = \frac{\mu_0}{4\pi}\int \frac{I(t - r/c)d\vec{l} \times \hat{r}}{r^2}.$$

$$B_\phi(r,\theta,t)=\left(\frac{\mu_0 I(t-r/c)\sin\theta}{4\pi r^2}\right)2s=-\frac{\mu_0\omega Q_0\sin(\omega t-kr)2s\sin\theta}{4\pi r^2}=-\frac{\mu_0\omega p_0\sin(\omega t-kr)\sin\theta}{4\pi r^2}$$

where $p_0 = 2Q_0 s$.

(c) However, Biot-Savart's law is only applicable to steady state currents. It is incomplete even after including the retarded nature of the oscillating current. By considering the wave nature of the magnetic field, the complete expression of the magnetic field is given by
但是，毕奥 - 萨伐尔定律只适用于稳态电流。甚至考虑了振动电流的滞后性质后，它还是不完整的。通过考虑磁场的波动性，完整的磁场表达式是

$$\mathbf{B}(r,\theta,t)=\mathbf{B}_{\mathrm{BS}}(r,\theta,t)+\mathbf{B}_{\mathrm{wave}}(r,\theta,t),$$

where 其中

$$\mathbf{B}_{\mathrm{wave}}(r,\theta,t)=\frac{\mu_0}{4\pi c}\int\left[\frac{d}{dt}I\left(t-\frac{r}{c}\right)\right]\frac{d\vec{l}\times\hat{r}}{r}.$$

Derive an expression for the B_ϕ component of $\mathbf{B}_{\mathrm{wave}}$ at (r, θ, t). (3 points)
试推导 $\mathbf{B}_{\mathrm{wave}}$ 在 (r, θ, t)的 B_ϕ 分量的表达式。（3 分）

$$B_{\mathrm{wave}}(r,\theta,t)=\frac{\mu_0}{4\pi c}\int\frac{d}{dt}\left(-\omega Q_0\sin(\omega t-kr)\right)\frac{\sin\theta}{r}dl=-\frac{\mu_0}{4\pi c}\left(\omega^2 Q_0\cos(\omega t-kr)\frac{\sin\theta}{r}\right)2s$$
$$=-\frac{\mu_0\omega^2 p_0}{4\pi cr}\cos(\omega t-kr)\sin\theta.$$

(d) Compare the amplitudes of B_{BS} and B_{wave} at large distance r. Derive the condition of r such that B_{BS} becomes negligible when compared with B_{wave}. (2 points)
比较 B_{BS} 和 B_{wave} 在距离 r 很大时的幅度。试推导 B_{BS} 相比 B_{wave} 变得微不足道时，关于 r 的条件。（2 分）

$$\frac{|B_{\mathrm{BS}}(r,\theta,t)|}{|B_{\mathrm{wave}}(r,\theta,t)|}=\frac{\mu_0\omega p_0\sin\theta/4\pi r^2}{\mu_0\omega^2 p_0\sin\theta/4\pi cr}=\frac{c}{\omega r}<<1$$
$$\Rightarrow\quad r>>\frac{c}{\omega}=\frac{\lambda}{2\pi}.$$

(e) At large distance r, the electric field at (r, θ, t) is mainly due to the electromagnetic induction by the magnetic field B_{wave}. By considering the electromotive force along the circuit shown in figure (B), derive the relation between $\frac{\partial E_\theta}{\partial r}$ and $\frac{\partial B_\phi}{\partial t}$. Here, $\frac{\partial E_\theta}{\partial r}$ is known as the partial derivative of E_θ with respect to r, meaning that other variables such as θ and t are considered fixed. Similarly, $\frac{\partial B_\phi}{\partial t}$ is the partial derivative of B_ϕ with respect to t, with other variables such as r and θ being fixed. You may assume that only the E_θ component of the electric field is significant at large distance r. (3 points)
在距离 r 很大时，在(r, θ, t)的电场主要是源于 B_{wave} 的电磁感应。通过考虑沿著图（B）中闭路的电动势，试推导 $\frac{\partial E_\theta}{\partial r}$ 与 $\frac{\partial B_\phi}{\partial t}$ 之间的关系。这里，$\frac{\partial E_\theta}{\partial r}$ 被称为 E_θ相对于 r 的偏

导数，意味着其他变量如θ和 t 被假定为固定的。同样地，$\frac{\partial B_\phi}{\partial t}$ 是 B_ϕ相对于 t 的偏导数，当中假定其他变量如 r 和θ 为固定的。你可以假设在距离 r 很大时，电场仅有 E_θ分量是显著的。（3 分）

Consider the electromotive force along the circuit. Total electromotive force:
考虑沿著闭路的电动势。总电动势：
$$\text{emf} = -E_\theta(r+dr)(r+dr)d\theta + E_\theta(r)rd\theta.$$
Magnetic flux enclosed by the circuit: 闭路的磁通量：$\Phi = (-B_\phi)(rd\theta dr)$.

Using Faraday's law, 利用法拉第定律， $\text{emf} = -\frac{d\Phi}{dt}$.
$$-E_\theta(r+dr)(r+dr)d\theta + E_\theta(r)rd\theta = \frac{\partial B_\phi}{\partial t} rd\theta dr.$$
$$-E_\theta(r+dr) - \frac{E_\theta(r)}{r}dr + E_\theta(r) = \frac{\partial B_\phi}{\partial t}dr.$$
In the limit dr approaches 0, 在 dr 趋近 0 时， $\frac{\partial E_\theta}{\partial r} + \frac{E_\theta}{r} = -\frac{\partial B_\phi}{\partial t}$.

(f) At large distance r, the electric field is given by $E_\theta(r,\theta,t) = \frac{A(\theta)}{r}\cos(\omega t - kr)$. Find $A(\theta)$. (2 points)

在距离 r 很大时，电场为 $E_\theta(r,\theta,t) = \frac{A(\theta)}{r}\cos(\omega t - kr)$。试找出 $A(\theta)$。（2 分）

$$\frac{\partial B_\phi}{\partial t} = \frac{\mu_0 \omega^3 p_0}{4\pi cr}\sin(\omega t - kr)\sin\theta.$$
$$\frac{\partial E_\theta}{\partial r} = -\frac{A(\theta)}{r^2}\cos(\omega t - kr) + \frac{kA(\theta)}{r}\sin(\omega t - kr) \approx \frac{kA(\theta)}{r}\sin(\omega t - kr).$$
$$\frac{E_\theta}{r} = \frac{A(\theta)}{r^2}\cos(\omega t - kr) << \frac{\partial E_\theta}{\partial r} \quad \Rightarrow \quad A(\theta) = -\frac{\mu_0 \omega^2 p_0}{4\pi}\sin\theta.$$

(g) The magnitude and direction of the power per unit area of the EM wave are given by the Poynting vector. Calculate the time-averaged power per unit area at large distance r. This will be denoted as the radiation intensity $I(r)$. (3 points)

电磁波每单位面积传播功率的大小和方向，是由 Poynting 矢量给定的。试计算在距离 r 很大时，每单位面积按时间平均的传播功率。这将被表示为辐射强度 $I(r)$。（3 分）

$$S(r) = \frac{1}{\mu_0}E_\theta B_\phi = \frac{1}{\mu_0}\left[-\frac{\mu_0\omega^2 p_0}{4\pi r}\cos(\omega t - kr)\sin\theta\right]\left[-\frac{\mu_0\omega^2 p_0}{4\pi cr}\cos(\omega t - kr)\sin\theta\right]$$
$$= \frac{\mu_0 \omega^4 p_0^2}{16\pi^2 r^2 c}\cos^2(\omega t - kr)\sin^2\theta.$$
$$I(r) = \langle S(r)\rangle = \frac{\mu_0\omega^4 p_0^2}{16\pi^2 r^2 c}\langle\cos^2(\omega t - kr)\rangle\sin^2\theta = \frac{\mu_0\omega^4 p_0^2}{32\pi^2 r^2 c}\sin^2\theta.$$

(h) When an EM wave is incident on a molecule, its electric field **E** will drive the molecule into an oscillating dipole moment given by **p** = α**E**, where α is the polarizability of the molecule.

In turn, the oscillating dipole will radiate power. This is called a scattering process. Consider an EM wave incident from the x direction, given by $\mathbf{E}_i = \mathbf{E}_{x0}\cos(\omega t - kx)$. If $\mathbf{E}_{x0}$ is polarized at an angle θ_x with the z axis, calculate:
当电磁波射向一分子时，其电场 **E** 会使该分子产生振动偶极矩 $\mathbf{p} = \alpha\mathbf{E}$，其中$\alpha$是该分子的极化度。随之振动偶极子会辐射功率。这就是所谓的散射过程。考虑电磁波从 x 方向入射，由 $\mathbf{E}_i = \mathbf{E}_{x0}\cos(\omega t - kx)$给出。若 $\mathbf{E}_{x0}$ 的偏振方向与 z 轴成角度θ_x，试计算：
(h1) the intensity $I_x(r)$ of the radiation scattered to the z direction,
散射至 z 方向的辐射强度 $I_x(r)$，
(h2) the electric field polarization of the scattered wave along that direction,
沿该方向的散射波的电场偏振方向，
(h3) the intensity $\langle I_x(r)\rangle$ of the radiation scattered to the z direction for an unpolarized incident beam (that is, the polarization angle θ_x has a uniform distribution). (3 points)
非偏振入射光束（即偏振角θ_x均匀分布）散射至 z 方向的辐射强度$\langle I_x(r)\rangle$。（3 分）

(h1) If $\mathbf{E}_{x0}$ is polarized at an angle θ_x with the z axis, then the dipole moment lies in the yz plane making an angle θ_x with the z axis. Its intensity is
若 $\mathbf{E}_{x0}$ 的偏振与 z 轴成角度θ_x，则偶极矩位于 yz 平面与 z 轴成角度θ_x。辐射强度为

$$I_x(r) = \frac{\mu_0\omega^4 p_0^2}{32\pi^2 r^2 c}\sin^2\theta_x = \frac{\mu_0\omega^4\alpha^2}{32\pi^2 r^2 c}E_{x0}^2\sin^2\theta_x .$$

(h2) The electric field of the scattered waves becomes polarized in the y direction.
散射波的电场偏振方向是 y 方向。

(h3) $$\langle I_x(r)\rangle = \frac{\mu_0\omega^4\alpha^2}{32\pi^2 r^2 c}E_{x0}^2\langle\sin^2\theta_x\rangle = \frac{\mu_0\omega^4\alpha^2}{64\pi^2 r^2 c}E_{x0}^2 .$$

(i) Next, consider an EM wave incident from the y direction, given by $\mathbf{E}_i = \mathbf{E}_{y0}\cos(\omega t - ky)$. If $\mathbf{E}_{y0}$ is polarized at an angle θ_y with the z axis, calculate:
接下来，考虑电磁波从 y 方向入射，由 $\mathbf{E}_i = \mathbf{E}_{y0}\cos(\omega t - ky)$给出。若 $\mathbf{E}_{y0}$ 的偏振与 z 轴成角度θ_y，试计算：
(i1) the electric field polarization of the scattered wave along the z direction,
沿 z 方向的散射波的电场偏振方向，
(i2) the intensity $\langle I_y(r)\rangle$ of the radiation scattered to the z direction for an unpolarized incident beam (that is, the polarization angle θ_y has a uniform distribution). (2 points)
非偏振入射光束（即偏振角θ_y均匀分布）散射至 z 方向的辐射强度$\langle I_y(r)\rangle$。（2 分）

(i1) The electric field of the scattered waves becomes polarized in the x direction.
散射波的电场偏振方向是 x 方向。
(i2) If $\mathbf{E}_{y0}$ is polarized at an angle θ_y with the z axis, then the dipole moment lies in the xz plane making an angle θ_y. Its intensity is
若 $\mathbf{E}_{y0}$ 的偏振与 z 轴成角度θ_y，则偶极矩位于 xz 平面与 z 轴成角度θ_y。辐射强度为

$$I_y(r) = \frac{\mu_0\omega^4 p_0^2}{32\pi^2 r^2 c}\sin^2\theta_y = \frac{\mu_0\omega^4\alpha^2}{32\pi^2 r^2 c}E_{y0}^2\sin^2\theta_y .$$

$$\langle I_y(r)\rangle = \frac{\mu_0\omega^4\alpha^2}{32\pi^2 r^2 c}E_{y0}^2\langle\sin^2\theta_y\rangle = \frac{\mu_0\omega^4\alpha^2}{64\pi^2 r^2 c}E_{y0}^2 .$$

(j) During the rapid expansion of the early universe, gravitational waves are formed. They consist of *quadrupolar* temperature oscillations, meaning that the directions of the maxima and minima of the oscillations are separated by an angle of $\pi/2$. Hence to analyze their effects on EM waves, we consider two incoherent incident beams of EM waves of the same frequency $\omega/2\pi$, one from the x direction and the other from the y direction. The amplitudes of their electric fields are E_{x0} and E_{y0} respectively. Suppose the EM radiations in the x and y directions correspond to temperatures $T + \Delta T$ and T respectively ($\Delta T << T$ and is positive). What is the ratio $\dfrac{\langle I_x(r)\rangle}{\langle I_y(r)\rangle}$? (1 point)

早期宇宙的迅速膨胀，形成引力波。它引起温度的振动，呈*四偶极*分布。这意味着振动的最大值和最小值的方向以$\pi/2$ 角度分开。因此，要分析它们对电磁波的影响，我们考虑两束频率同为$\omega/2\pi$的非相干入射光，一束来自 x 方向，另一束则来自 y 方向，其电场的幅度分别是 E_{x0} 和 E_{y0}。假设在 x 和 y 方向的电磁辐射分别对应于温度 $T + \Delta T$ 和 T（$\Delta T << T$，且是正的）。比例 $\dfrac{\langle I_x(r)\rangle}{\langle I_y(r)\rangle}$ 是什么？（1 分）

$$\frac{\langle I_x(r)\rangle}{\langle I_y(r)\rangle} = \frac{T+\Delta T}{T}.$$

(k) The degree of polarization of the scattered radiation is given by 下式是散射辐射的偏振度

$$\Pi = \frac{\left|\langle I_x(r)\rangle - \langle I_y(r)\rangle\right|}{\langle I_x(r)\rangle + \langle I_y(r)\rangle}.$$

Calculate Π. What is the direction of the electric field polarization in the scattered wave? (2 points)

试计算Π。散射辐射中电场的偏振方向是什么？（2 分）

$$\Pi = \frac{\left|\langle I_x(r)\rangle - \langle I_y(r)\rangle\right|}{\langle I_x(r)\rangle + \langle I_y(r)\rangle} = . \ \Pi = \frac{\left|\langle I_x(r)\rangle - \langle I_y(r)\rangle\right|}{\langle I_x(r)\rangle + \langle I_y(r)\rangle} = \frac{T+\Delta T - T}{T+\Delta T + T} = \frac{\Delta T}{2T+\Delta T} \approx \frac{\Delta T}{2T}.$$

Since $\langle I_x(r)\rangle$ is stronger, the electric field polarization in the scattered radiation is the y direction.

《THE END 完》