Pan Pearl River Delta Physics Olympiad 2016
2016 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
香港科技大学高等研究院赞助
Simplified Chinese Part-1 (Total 5 Problems) 简体版卷-1（共5题）
(9:00 am – 12:00 pm, 18 February, 2016)

1. Electrostatic Force (4 marks) 静电力（4分）

Consider a 2017-side regular polygon. There are 2016 point charges, each with charge q and located at a vertex of the polygon. Another point charge Q is located at the center of the polygon. The distance from the center of the regular polygon to its vertices is a. Find the force experienced by Q.
考虑一 2017-边正多边形。其中 2016 个角上各有一点电荷 q。 另有一个点电荷 Q 位于多边形的中心。 中心到每一个角的距离为 a。 求 Q 所受的力。

Consider the polygon with a charge q at each vertex. In other words, there are 2017 charges. The system has a discrete rotational symmetry and hence the force acting on Q must be zero. Now our system is equivalent to the above system but with a charge $-q$ added to one vertex. Hence the force is

$$\mathbf{F} = \frac{Qq}{4\pi\varepsilon_0 a^2}\hat{\mathbf{a}}$$

where $\hat{\mathbf{a}}$ is a unit vector pointing from the center to the empty vertex.

2. Capacitors (13 marks) 电容器（13 分）

(a-c) Consider two clusters of electric charges. Cluster A consists of N charges $q_1, q_2, \ldots, q_N$, located at positions $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N$ respectively. Cluster B consists of M charges $q_1', q_2', \ldots, q_M'$, located at positions $\vec{r}_1', \vec{r}_2', \ldots, \vec{r}_M'$ respectively.
(a-c) 考虑两组电荷。组 A 由 N 个电荷 $q_1, q_2, \ldots, q_N$ 组成, 并分别位于位置$\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N$。组 B 由 M 个电荷$q_1', q_2', \ldots, q_M'$组成, 并分别位于位置$\vec{r}_1', \vec{r}_2', \ldots, \vec{r}_M'$。

(a) Write the electric potential $\phi_A(\vec{r})$ at position $\vec{r}$ due to the charges in cluster A. (1 mark)
写下于位置$\vec{r}$ 由组 A 电荷形成的电势$\phi_A(\vec{r})$。（1 分）

$$\phi_A(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i}{|\vec{r}-\vec{r}_i|}.$$

(b) Write the electric potential energy $E_{B|A}$ of cluster B due to the electric potential ϕ_A. (1 mark)
写下组 B 电荷因电势ϕ_A产生的电势能 $E_{B|A}$。 (1 分)

$$E_{B|A} = \sum_{i=1}^{M} q_i'\phi_A(\vec{r}_i') = \sum_{i=1}^{M} q_i'\left(\frac{1}{4\pi\varepsilon_0}\sum_{j=1}^{N}\frac{q_j}{|\vec{r}_i'-\vec{r}_j|}\right) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{M}\sum_{j=1}^{N}\frac{q_i' q_j}{|\vec{r}_i'-\vec{r}_j|}.$$

(c) What is the relation between $E_{B|A}$ and $E_{A|B}$? (1 mark)
$E_{B|A}$ 和 $E_{A|B}$有何关系? (1 分)

$$E_{A|B} = \sum_{i=1}^{N} q_i \phi_B(\vec{r}_i) = \sum_{i=1}^{N} q_i \left(\frac{1}{4\pi\varepsilon_0} \sum_{j=1}^{M} \frac{q_j'}{|\vec{r}_i - \vec{r}_j'|} \right) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \sum_{j=1}^{M} \frac{q_i q_j'}{|\vec{r}_i - \vec{r}_j'|}.$$

Interchanging the indices i and j, and the order of summation, we have $E_{B|A} = E_{A|B}$.

(d) Consider two large conducting plates as shown in Fig. 1a. The upper plate carries a uniform surface charge density σ' and the lower plate is grounded. Find the surface charge density of the lower plate and the potential $\phi'(z)$, where z is the height of an arbitrary location from the lower plate. (5 marks)

考虑如图 1a 所示两块很大的电导板。上板带有均匀面电荷密度σ'，而下板则接地。求下板的面电荷密度和电势$\phi'(z)$，其中z为任意一点距离下板的高度。（5 分）

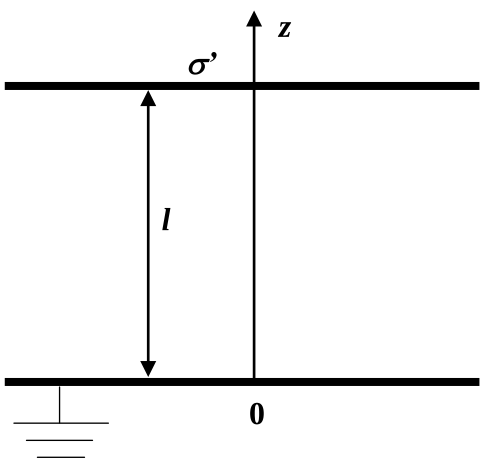

Figure 1a 图 1 a

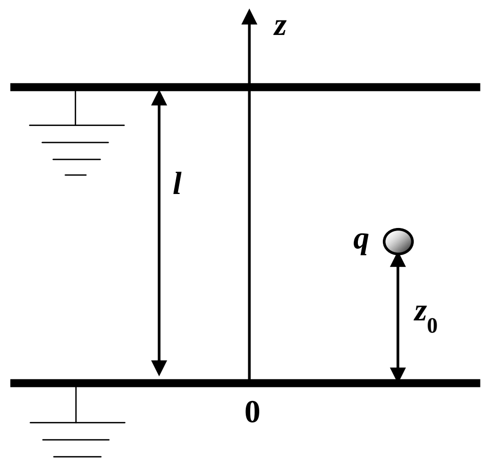

Figure 1b 图 1b

The surface charge density at the lower plate is $-\sigma'$.

Using Gauss' law, the electric field between the plates is $E = \frac{\sigma'}{\varepsilon_0}$.

The potential is $\phi'(z) = \begin{cases} 0 & z \le 0, \\ \frac{\sigma'}{\varepsilon_0} z & 0 \le z \le l, \\ \frac{\sigma'}{\varepsilon_0} l & z > l. \end{cases}$

(e) A point charge q is placed between two very large grounded parallel conducting plates. If z_0 is the distance between q and the lower plate, find the total charge induced on the upper plate in terms of q, z_0, and l, where l is the distance between the plates, as shown in Fig. 1b. (5 marks)

如图 1b 所示，在相距为 l 的两块平行大电导板间放置电荷 q，其到下板的距离为 z_0。求上板的总感应电荷。以 q、z_0 和 l 表示你的答案。（5 分）

Consider the charge distribution in Fig. 1a to be cluster A, and that in Fig. 1b to be cluster B.

To calculate $E_{A|B}$, we note that there are electric charges in cluster A located at the upper plate only, but for cluster B, the electric potential at the upper plate is 0. Hence $E_{A|B} = 0$.

To calculate $E_{B|A}$, we note that there are electric charges in cluster B located at:

- the lower plate, but $\phi_A = 0$;
- the point charge q, where $\phi_A = \frac{\sigma'}{\varepsilon_0} z_0$;

- the upper plate with charge Q_u to be determined, where $\phi_A = \frac{\sigma'}{\varepsilon_0} l$.

Hence applying the result in part (c), $q\frac{\sigma'}{\varepsilon_0} z_0 + Q_u \frac{\sigma'}{\varepsilon_0} l = 0 \Rightarrow Q_u = -q\frac{z_0}{l}$

3. Cannonballs and Bombs (10 marks) 砲彈和炸彈（10 分）

(a) Envelope of safety: A ground based cannon can fire a cannonball at a fixed speed of u in any direction. The envelope of safety is the curve inside which a target can be hit by the cannonball, and outside which there is no possibility of a target getting hit by the cannonball. Find the equation of the envelope of safety in space. (3 marks)
安全区域边界：一门位于地面的大炮能以固定速率 u 向任何方向发射炮弹。 若目标在安全区域边界内，则有可能被炮弹打中。 若在其外，则不可能被炮弹打中。 求在空中的安全区域边界的方程式。（3 分）

Consider a target at (x, y). Let the cannonball fired at angle θ hit this point. Then

$$\begin{cases} x = ut\cos\theta, \\ y = ut\sin\theta - \frac{1}{2}gt^2. \end{cases}$$

Eliminating t, $y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$.

Note that $\frac{1}{\cos^2\theta} = \sec^2\theta = 1 + \tan^2\theta$, we arrive at the quadratic equation in $\tan\theta$.

$$\frac{gx^2}{2u^2}\tan^2\theta - x\tan\theta + \frac{gx^2}{2u^2} + y = 0.$$

Outside the envelope, there are no solutions.
Inside the envelope, there are two solutions.
Right on the envelope, there is only one solution.
Setting the discriminant to zero, $\Delta = (-x)^2 - 4\frac{gx^2}{2u^2}\left(\frac{gx^2}{2u^2} + y\right) = 0 \Rightarrow y = -\frac{g}{2u^2}x^2 + \frac{u^2}{2g}$ which is a parabola.

(b) A bomb explodes at a height of H into many small fragments. It is given that after the explosion the fragments have the same speed u and a uniform angular distribution in all directions. After some time all fragments hit the ground and the collisions with the ground are perfectly inelastic. Find the radius R of the distribution of the debris. (2 marks)
一个炸弹在高度 H 处爆炸成很多小碎片。 已知刚爆炸后各碎片以同样的速率 u 和均匀的角分布向各个方向散开。 其后各碎片都坠到地面上。 假设所有碎片与地面的碰撞皆为完全非弹性碰撞。 求炸弹残骸的分布半径 R。（2 分）

Let the bomb be located at the origin, x be the horizontal axis and y be the vertical axis, with upward as positive. Using the result of (a), set $y = -H$, we have $-H = -\frac{g}{2u^2}R^2 + \frac{u^2}{2g} \Rightarrow$

$$R = \frac{u}{g}\sqrt{u^2 + 2gH}.$$

(c) A bomb explodes on the ground. Its fragments are projected at the same speed u, and the angular distribution is uniform within a narrow angle α with the upward vertical direction. After some time all fragments hit the ground. Let the mass of the bomb be M. Find the radius R of the distribution of the debris up to order α. Calculate the radial density distribution $\rho(r)$

within radius R up to order r^2, where $\rho(r)2\pi r dr$ is the mass of the debris located at a distance r to $r + dr$ from the centre of the distribution. (5 marks)

[Remark: $\tan\varepsilon \approx \varepsilon\left(1 + \frac{\varepsilon^2}{3}\right)$ and $\sin\varepsilon \approx \varepsilon\left(1 - \frac{\varepsilon^2}{6}\right)$ for $\varepsilon \ll 1$.]

一个炸弹在地面爆炸。爆炸后各碎片以同样速率 u 射出，角度分布则限在与垂直向上方向的狭小夹角α内，而在这范围内角度分布均匀。其后各碎片都坠到地面上。设炸弹的质量为 M。求炸弹残骸的分布半径 R, 准确至α 的第一阶。定义径密度分布$\rho(r)$，使得$\rho(r)2\pi r dr$为距离残骸中心 r 至 r+dr 范围内的残骸质量。求半径 R 内的$\rho(r)$, 准确至 r 的第二阶。（5 分）

[注: 当 $\varepsilon \ll 1$ 时, $\tan\varepsilon \approx \varepsilon\left(1 + \frac{\varepsilon^2}{3}\right)$及$\sin\varepsilon \approx \varepsilon\left(1 - \frac{\varepsilon^2}{6}\right)$。]

$$\begin{cases} x = ut\cos\theta, \\ y = ut\sin\theta - \frac{1}{2}gt^2. \end{cases}$$

At $y = 0$, $ut\sin\theta - \frac{1}{2}gt^2 = 0 \Rightarrow t = \frac{2u\sin\theta}{g} \Rightarrow x = u\cos\theta\left(\frac{2u\sin\theta}{g}\right) = \frac{u^2\sin 2\theta}{g}$

Substituting $\theta = \frac{\pi}{2} - \varepsilon$ and $x = r$,

$$r = \frac{u^2}{g}\sin(\pi - 2\varepsilon) = \frac{u^2}{g}\sin(2\varepsilon) \approx \frac{2u^2}{g}\varepsilon\left(1 - \frac{2}{3}\varepsilon^2\right).$$

Hence $R \approx \frac{2u^2}{g}\alpha$.

The angular distribution after the explosion is $\rho(\theta)d\theta = M\frac{2\pi\cos\theta d\theta}{2\pi(1-\cos\alpha)} \Rightarrow \rho(\theta) \approx \frac{2M}{\alpha^2}\cos\theta$

$\Rightarrow \rho(\varepsilon) = \frac{2M}{\alpha^2}\cos\left(\frac{\pi}{2} - \varepsilon\right) = \frac{2M}{\alpha^2}\sin\varepsilon \approx \frac{2M}{\alpha^2}\varepsilon\left(1 - \frac{\varepsilon^2}{6}\right)$.

$\rho(r) = \frac{\rho(\varepsilon)}{2\pi r}\frac{d\varepsilon}{dr}$.

$\frac{dr}{d\varepsilon} \approx \frac{2u^2}{g}(1 - 2\varepsilon^2) \Rightarrow \rho(r) \approx \frac{Mg}{2\pi r u^2\alpha^2}\varepsilon\left(1 + \frac{11\varepsilon^2}{6}\right) \approx \frac{Mg^2}{4\pi u^4\alpha^2}\left(1 + \frac{11\varepsilon^2}{6}\right)\left(1 + \frac{2\varepsilon^2}{3}\right)$

$\approx \frac{Mg^2}{4\pi u^4\alpha^2}\left(1 + \frac{5\varepsilon^2}{2}\right) \approx \frac{Mg^2}{4\pi u^4\alpha^2}\left(1 + \frac{5g^2r^2}{8u^4}\right)$.

4. Collisions (14 marks) 碰撞 (14 分)

A thin rod with length L, mass m and uniform density lies on the y-axis with its midpoint at the origin. A point object A with mass m travels with velocity u in the positive x direction hits the rod with impact parameter h, where $-L/2 \le h < L/2$, as shown in Fig. 2a. The collision is perfectly inelastic.

如图 2a 所示, 一根长度为 L、质量为 m、密度均匀的幼棒处在 y 轴上。 棒的中心点位于原点。 一质量为 m 的质点 A 以速度 u 向正 x 方向运动, 并以碰撞参数 h 与棒碰撞, 其中$-L/2 \le h < L/2$,。碰撞为完全非弹性碰撞。

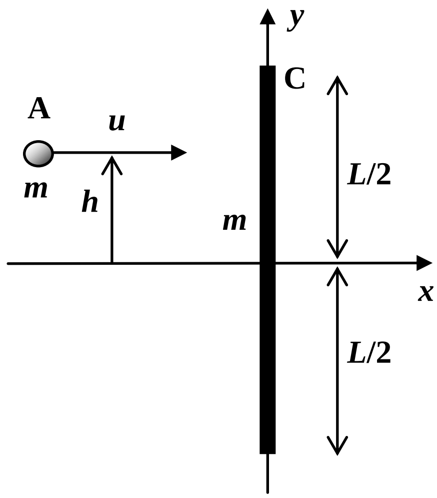

Figure 2a 图 2a

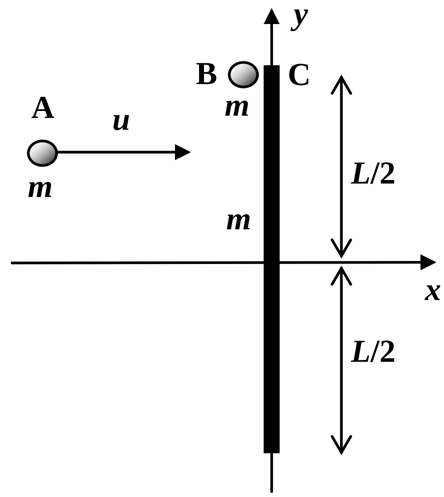

Figure 2b 图 2b

(a) Find the total kinetic energy just after the collision between A and the rod. (6 marks)
求 A 刚与棒碰撞后系统的总动能。（6 分）

Initial momentum: $p_i = mu$

Initial angular momentum about origin (clockwise positive): $L_i = muh$

Initial kinetic energy: $K_i = \frac{1}{2}mu^2$

Final momentum: $p_f = 2mv_{CM}$

Final angular momentum about origin: $L_f = 2mv_{CM}y_{CM} + I_{CM}\omega$ where $y_{CM} = \frac{m}{m+m}h = \frac{h}{2}$.

$$I_{CM} = \frac{1}{12}mL^2 + m\left(\frac{h}{2}\right)^2 + m\left(h - \frac{h}{2}\right)^2 = \frac{1}{12}mL^2 + \frac{1}{2}mh^2$$

Using the conservation of momentum, $mu = 2mv_{CM} \Rightarrow v_{CM} = \frac{u}{2}$.

Using the conservation of angular momentum,

$muh = (2m)\left(\frac{u}{2}\right)\frac{h}{2} + \left(\frac{1}{12}mL^2 + \frac{1}{2}mh^2\right)\omega \Rightarrow \omega = \frac{6uh}{L^2+6h^2}$.

Final kinetic energy:

$$K_f = \frac{1}{2}(2m)\left(\frac{u}{2}\right)^2 + \frac{1}{2}\left(\frac{1}{12}mL^2 + \frac{1}{2}mh^2\right)\left(\frac{6uh}{L^2 + 6h^2}\right)^2 = \frac{mu^2}{4}\left(1 + \frac{6h^2}{L^2 + 6h^2}\right)$$

(b) Find the velocity v of point C at the top end of the rod as a function of h. (2 marks)
求棒上端点 C 的速度 v 与 h 的函数关系。（2 分）

$$v(h) = \frac{u}{2} + \frac{6uh}{L^2 + 6h^2}\left(\frac{L}{2} - \frac{h}{2}\right) = \frac{uL(L + 6h)}{2(L^2 + 6h^2)}.$$

(c) Find H such that $v(H) = 0$. (1 mark)
求 H 使得 $v(H) = 0$。（1 分）

$v(H) = \frac{uL(L+6H)}{2(L^2+6H^2)} = 0 \Rightarrow H = -\frac{L}{6}$.

(d) Suppose another point object B of mass m is located very close to point C, at the left hand side, as shown in Fig. 2b. Further suppose the point object A hits the rod at the lower end. Find the velocity of the point object B just after the rod collides elastically with it. (5 marks)
假设另一质量为 m 的质点 B 的位置与棒顶端 C 的左边非常接近，如图 2b 所示。再设点 A 撞到棒的下端。求棒与质点 B 产生完全弹性碰撞后，质点 B 的速度。（5 分）

Let w_1 be the forward velocity of the center of mass of the rod after collision with B.
Let w_2 be the backward velocity of object B after the collision.
Let ω_1 be the clockwise angular velocity of the rod after collision with B.
Conservation of linear momentum: $mu = -mw_2 + 2mw_1 \Rightarrow w_2 = 2w_1 - u$.
Conservation of angular momentum about the origin: $muh = -mw_2 \frac{L}{2} + I_{CM}\omega_1 + 2mw_1 \left(\frac{h}{2}\right)$.
Since $I_{CM} = \frac{1}{12}mL^2 + \frac{1}{2}mh^2 = \frac{5}{24}mL^2$, this implies $-\frac{1}{2}u = -\frac{1}{2}w_2 + \frac{5}{24}L\omega_1 - \frac{1}{2}w_1$.
Eliminating w_2, $\omega_1 = \frac{12}{5L}(3w_1 - 2u)$.
Conservation of energy: $\frac{mu^2}{4}\left(1 + \frac{6h^2}{L^2+6h^2}\right) = \frac{1}{2}mw_2^2 + \frac{1}{2}I_{CM}\omega_1^2 + \frac{1}{2}(2m)w_1^2 \Rightarrow$
$\frac{2}{5}u^2 = \frac{1}{2}w_2^2 + \frac{5}{48}L^2\omega_1^2 + w_1^2$.
Substituting w_2 and ω_1, $\frac{2}{5}u^2 = \frac{1}{2}(2w_1 - u)^2 + \frac{3}{5}(3w_1 - 2u)^2 + w_1^2$.
This reduces to the quadratic equation $84w_1^2 - 92uw_1 + 25u^2 = 0 \Rightarrow w_1 = \frac{25}{42}u$ or $\frac{u}{2}$. The second solution is the velocity before collision. Hence $w_1 = \frac{25}{42}u \Rightarrow w_2 = \frac{4}{21}u$.

5. Thermodynamic Cycle (9 marks) 热力学循环 (9 分)

Consider the thermodynamic cycle of an ideal monatomic gas shown in the pV diagram in Fig. 3. The cycle consists of four processes:

- A → B: Isobaric expansion at pressure rp, where $r > 1$
- B → C: Isothermal expansion at temperature T_2
- C → D: Isobaric compression at pressure p
- D → A: Isothermal compression at temperature T_1

考虑图 3 中所示一种单原子理想气体的热力学循环的 pV 图。 该循环包括四个过程:

- A → B: 压强 rp 下的等压膨胀, 其中 $r > 1$
- B → C: 温度 T_2 下的等温膨胀
- C → D: 压强 p 下的等压压缩
- D → A: 温度 T_1 下的等温压缩

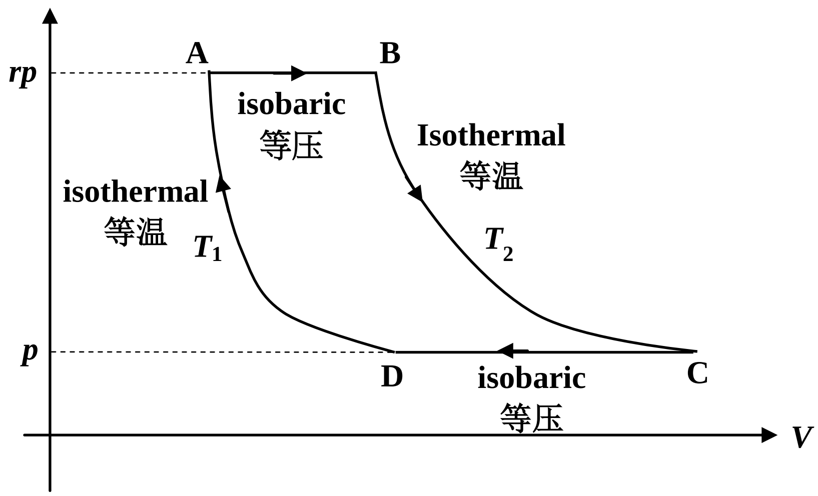

Figure 3 图 3

(a) Write the highest temperature T_H and lowest temperature T_C in the cycle. No proof is required. (1 mark)
无须证明, 写下循环中的最高温度 T_H 和最低温度 T_C。（1 分）

$T_H = T_2$, $T_C = T_1$.

(b) Write the efficiency e_C of a Carnot engine operating with a hot reservoir at temperature T_H and a cold reservoir at temperature T_C. (1 mark)
一卡诺热机在温度为 T_H 的高温热库和温度为 T_C 的低温热库间运行。 写下其热效率 e_C。(1 分)

$e_C = 1 - \frac{T_C}{T_H}$.

(c) Given that the gas is in thermal contact with a hot reservoir with temperature T_H whenever heat is added to the gas, and in thermal contact with a cold reservoir with temperature T_C whenever heat is removed from the gas, find the efficiency e of an engine running the cycle in the pV diagram. Express your answer in terms of T_C, T_H, p, and r. (5 marks)
已知一热机在循环运行中，气体吸热时永远与温度为 T_H 的热库处于热接触, 气体放热时永远与温度为 T_C 的热库处于热接触。求其热效率 e。 以 T_C、T_H、p 和 r 表示你的答案。（5 分）

In A → B

$$Q = \Delta U - W = \frac{3}{2}nR(T_H - T_C) + rp(V_B - V_A) = \frac{3}{2}nR(T_H - T_C) + nR(T_H - T_C)$$
$$= \frac{5}{2}nR(T_H - T_C)$$

In B → C

$$Q = -W = \int_{V_B}^{V_C} pdV = \int_{V_B}^{V_C} \frac{nRT_H}{V}dV = nRT_H \ln\frac{V_C}{V_B} = nRT_H \ln\frac{\frac{nRT_H}{p}}{\frac{nRT_H}{rp}} = nRT_H \ln r$$

In C → D

$$Q = \Delta U - W = \frac{3}{2}nR(T_C - T_H) - p(V_C - V_D) = \frac{3}{2}nR(T_C - T_H) - nR(T_H - T_C) =$$

$$= -\frac{5}{2}nR(T_H - T_C)$$

In D → A

$$Q = -W = \int_{V_D}^{V_A} p dV = \int_{V_D}^{V_A} \frac{nRT_C}{V} dV = -nRT_C \ln\frac{V_D}{V_A} = -nRT_C \ln\frac{\frac{nRT_D}{p}}{\frac{nRT_A}{rp}} = -nRT_C \ln r$$

Since $\Delta U = 0$ in a cycle, work done in a cycle = heat absorbed in a cycle
$= Q_{AB} + Q_{BC} + Q_{CD} + Q_{DA} = nR(T_H - T_C)\ln r$.

Heat is input during A → B and B → C, $Q_{AB} + Q_{BC} = \frac{5}{2}nR(T_H - T_C) + nRT_H \ln r$.

The efficiency is

$$e = \frac{nR(T_H - T_C)\ln r}{\frac{5}{2}nR(T_H - T_C) + nRT_H \ln r} = \frac{(T_H - T_C)\ln r}{\frac{5}{2}(T_H - T_C) + T_H \ln r}$$

(d) Find the ratio $\frac{e}{e_C}$. Hence suggest a parameter regime in which the efficiency approaches that of the ideal engine. (2 marks)

求比例 $\frac{e}{e_C}$。根据答案, 提出能使热效率趋近理想热机热效率的参数范围。 (2 分)

$$\frac{e}{e_C} = \left[\frac{(T_H - T_C)\ln r}{\frac{5}{2}(T_H - T_C) + T_H \ln r}\right]\left(\frac{T_H}{T_H - T_C}\right) = \frac{T_H \ln r}{\frac{5}{2}(T_H - T_C) + T_H \ln r} < 1.$$

To make the ratio approaches 1, we can make $r \gg 1$ or make $T_H - T_C \ll T_H$.

《THE END 完》