**Pan Pearl River Delta Physics Olympiad 2016**
2016 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
香港科技大学高等研究院赞助
**Simplified Chinese Part-2 (Total 3 Problems) 简体版卷-2（共3题）**
2:00 pm – 5:00 pm, 18 February, 2016

## 1. Gravitational Lens Simulator (9 marks) 引力透镜仿真镜 (9 分)

The gravitational field of a massive body exercises a lenslike condensing action upon radiation passing through it. A simulated gravitational lens was constructed of Plexiglas for use in demonstrating the lens phenomenon.
大质量物体附近的引力场会对经过的光线产生类似透镜的作用, 称为引力透镜效应。 这里我们考虑以有机玻璃制造一个光学透镜以仿真引力透镜现象。
According to general relativity, a light ray passing at a distance of closest approach $r$ to the center of a spherically symmetric body of mass $M$, will be deflected toward the body through an angle which, for small deflections, is given by
根据广义相对论, 当一束光线经过一个拥有球对称、质量为$M$的物体时,会产生屈折。当屈折角度很小，而光线与物体中心最靠近距离为 $r$ 时,该角度可由下面的公式得出

$$\varepsilon = \frac{4GM}{rc^2}$$

where $G$ is the gravitational constant and $c$ is the speed of light. We, therefore, require of our simulator that it deflects transmitted light through an angle
在上式中$G$为引力常数， $c$为光速。 因此我们要求仿真镜以下式中的角度屈折光线

$$\varepsilon = \frac{R}{r}$$

where $R$ is a constant. 在上式中$R$是某个常数。
The lens, illustrated in cross section in Fig. 1, is designed to be hand held within the range between roughly one foot and arm's length from the observer. The object for viewing is assumed to be at a distance much larger than one meter on the left hand side of the lens. This implies that one can assume the incident light ray to be normal to the plane front surface of the lens and refraction is thus assumed to take place entirely at the back surface. The angle of incidence of the light ray at the back surface is designated by $\theta$, the angle of refraction by $\theta$', and the angle of deflection by $\varepsilon$, which are all assumed to be small angles. The refractive index of the lens is $n$.
图一所示为仿真镜的横截面。仿真镜为观察者手举而设计，设计距离为大约一呎到手臂长度。观察对象位于图中仿真镜左边远大于一米处。因此我们可以假设入射光是垂直于仿真镜的前平面, 而折射仅发生于仿真镜的后表面。 以$\theta$ 表示入射光与后表面法线的夹角, $\theta$' 表示折射角度，$\varepsilon$表示屈折角度 。以上角度皆假设为小角度。仿真镜的折射系数是 $n$。

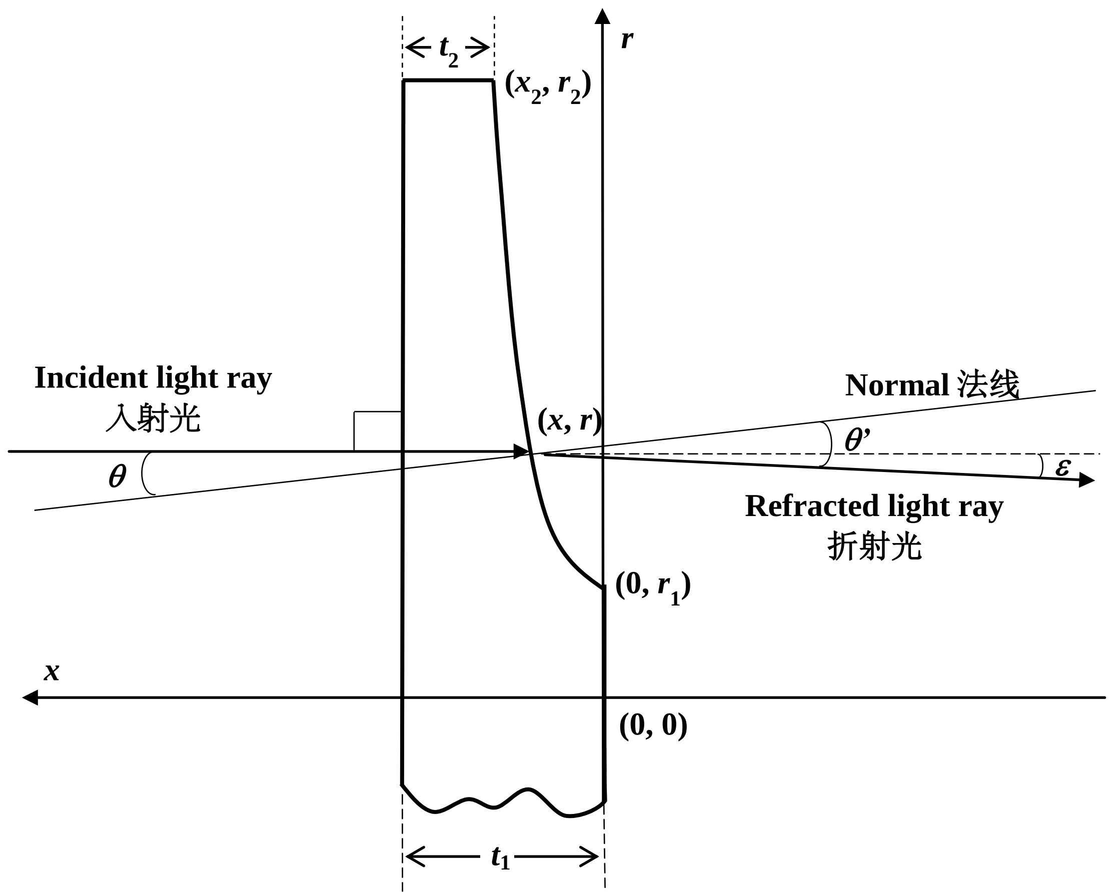



Figure 1 图一

(a) Derive the expression $dx/dr$ for the slope of the back refracting surface of the lens in terms of $\varepsilon$ and $n$. (4 marks)
试推导透镜后折射面的斜率 $dx/dr$，答案以$\varepsilon$和 $n$ 表达。(4 分)

Using Snell's law, $n \sin\theta = \sin\theta'$
By small angle assumption, $n\theta = \theta'$
The angle of deflection is thus $\varepsilon = \theta' - \theta = (n-1)\theta$
We also have $\tan(-\theta) = -\frac{1}{dr/dx} \Rightarrow \frac{dx}{dr} = \tan\theta$
By small angle assumption $\frac{dx}{dr} = \theta = \frac{\varepsilon}{n-1}$

(b) Derive the expression $x(r)$ of the back refracting surface of the lens. Express your answer in terms of $n$, $R$ and $r_1$. (3 marks)
试推导透镜后折射面的表达式 $x(r)$。答案以 $n$、$R$ 和 $r_1$ 表达。(3 分)

Using $\varepsilon = \frac{R}{r}$, we have $\frac{dx}{dr} = \frac{\varepsilon}{n-1} = \frac{R}{(n-1)r} \Rightarrow \; dx = \frac{Rdr}{(n-1)r}$.
Integrating, $x = \int \frac{R}{(n-1)r} dr = \frac{R}{n-1} \ln r + C$
Boundary condition: $0 = \frac{R}{n-1} \ln r_1 + C \Rightarrow C = -\frac{R}{n-1} \ln r_1$

$\Rightarrow x = \frac{R}{n-1}\ln r - \frac{R}{n-1}\ln r_1 \Rightarrow x = \frac{R}{n-1}\ln\frac{r}{r_1}$

(c) A light ray is incident at impact parameter $r$ equal to 2 cm. If we require the ray to cross the lens axis at a horizontal distance of 30 cm from the point at which it departs from the lens, find $R$. (1 mark)
考虑一束入射参数 $r$ 为 2 cm 的光线。 如果要求该光线在离开透镜后于水平距离 30 cm 处与透镜中轴相交, 则 $R$ 应取何值? (1 分)

$\tan\varepsilon = \frac{2}{30} \Rightarrow \varepsilon = \tan^{-1}\frac{2}{30} = \frac{R}{2} \Rightarrow R = 2\tan^{-1}\frac{2}{30} = 0.1331 \approx 0.133$ cm

Remark: Also accept answer using small angle approximation: $\varepsilon \approx \tan\varepsilon = \frac{2}{30} = \frac{R}{2} \Rightarrow R = \frac{2^2}{30} = 0.1333 \approx 0.133$ cm.

(d) Find the effective gravitational mass of the lens. (1 mark)
计算透镜的有效引力质量。(1 分)

$R = \frac{4GM}{c^2} \Rightarrow M = \frac{Rc^2}{4G} = \frac{(0.001331)(3\times10^8)^2}{(4)(6.67\times10^{-11})} = 4.489\times10^{23} \approx 4.49\times10^{23}$kg
which is about 6 times the mass of the Moon.
Reference: S. Liebes Jr., Am. J. Phys. **37**, 103 (1969).

**2. A String and a Mass (19 marks) 系着质量的弦 (19 分)**

In this question, all oscillatory motions are assumed to be small.
在本题中，假设所有震荡皆为微小震荡。

As shown in Fig. 2, a string of mass $m$ and length $l$ with tension $\tau$ has a mass $M$ attached to the end. The mass $M$ can slide in a vertical direction on a frictionless rod at $x = l$. The shape of the string is described by a function $y(x, t)$. The string is fixed at the origin $y(0, t) = 0$.
如图二所示, 一条质量为 $m$、长度为 $l$、张力为$\tau$的弦一端系着质量 $M$。 该质量 $M$ 可在一根位于 $x = l$ 处的平滑棒上自由垂直滑动。 弦的型态由函数 $y(x, t)$给出。 弦的另一端固定于原点处, 即 $y(0, t) = 0$。

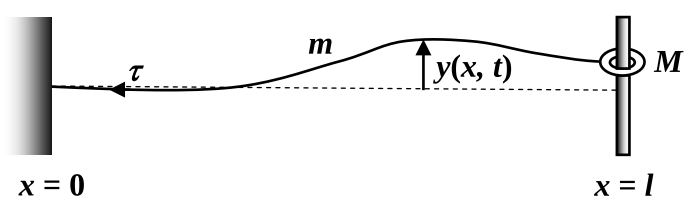



Figure 2 图二

(a) First assume that mass $M$ is held fixed at $y = 0$. Write down the general solution $y(x, t)$ for the standing waves on the string. Express your answer in terms of the given parameters and arbitrary constants. (4 marks)
首先假设质量 $M$ 被固定在 $y = 0$ 处。写下弦上驻波的一般解 $y(x, t)$。答案以给定的参数和任意常数表达。(4 分)

A standing wave has a general form $y(x,t) = (A\sin kx + B\cos kx)\sin(\omega t + \delta)$.

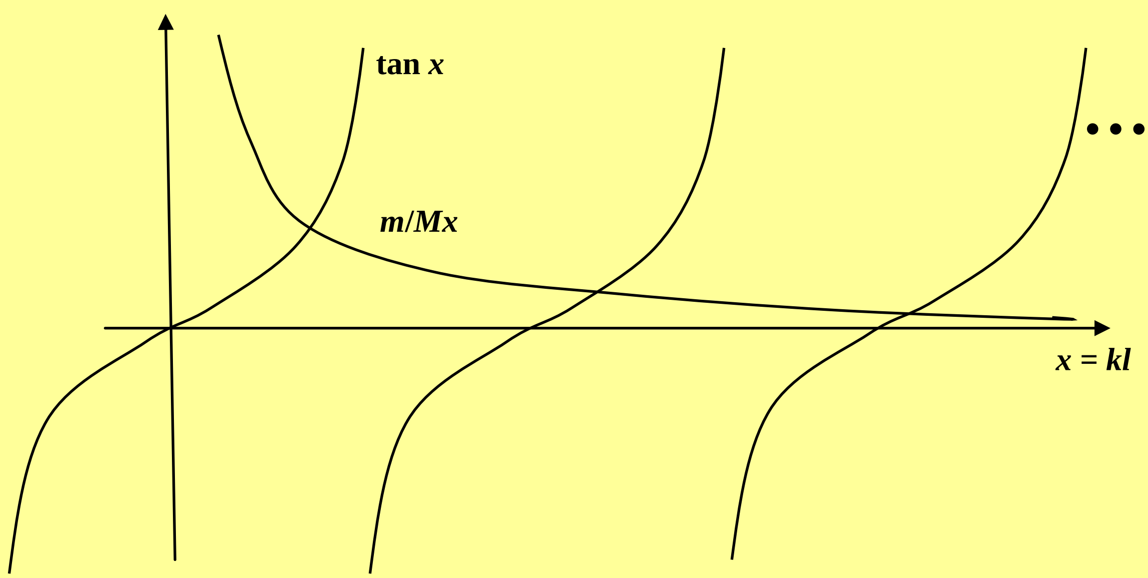
We can always choose the zero point of $t$ so that $\delta = 0$.
Hence $y(x,t) = (A\sin kx + B\cos kx)\sin\omega t$.
The boundary conditions $y(0, t) = 0$ and $y(l, t) = 0$ impose conditions $B = 0$ and $kl = n\pi$ for $n = 1, 2, 3, \ldots$
In general for a wave $\omega = kv$ and for a string $v = \sqrt{\tau/\rho} = \sqrt{l\tau/m}$. General solution:
$y(x,t) = \sum_{n=1}^{\infty} A_n \sin k_n x \sin \omega_n t$, where $k_n = \frac{n\pi}{l}$ and $\omega_n = \sqrt{\frac{l\tau}{m}} k_n$.

(b) Now assume that mass $M$ can slide up and down on a frictionless rod at $x = l$. What is the boundary condition on $y(x, t)$ at $x = l$? You can assume that the oscillations are small. (1 mark)
现在假设质量 $M$ 能沿着 $x = l$ 处的平滑棒上下移动，则 $y(x, t)$在 $x = l$ 处的边界条件该是甚么? 你可以假设震荡都是小震荡。(1 分)

The force needed to accelerate the mass is given by the tension in the string. Using Newton's law,
$$M\frac{\partial^2 y(l,t)}{\partial t^2} = -\tau \left.\frac{\partial y(x,t)}{\partial x}\right|_{x=l}$$

(c) Write down an equation for the frequencies of the standing waves on the string when the mass $M$ is free to slide. You do not need to solve the equation. (2 marks)
当质量 $M$ 可自由滑动时, 写下满足弦驻波频率的方程式。你不需要解该方程式。(2 分)

$$-M\omega^2 A\sin kl \sin\omega t = -\tau k A\cos kl \sin\omega t \Rightarrow Mk^2\frac{l\tau}{m}\sin kl = \tau k\cos kl \Rightarrow M\frac{kl}{m}\sin kl = \cos kl$$
$$kl\tan kl = \frac{m}{M}$$
There are infinitely many normal modes.




(d) If $m << M$, find the two lowest normal mode frequencies. (2 marks)
如果 $m << M$, 找出最低频的两个正则模频率。(2 分)

First normal mode:
Since $kl$ <<1, $\tan kl \approx kl$ and we have

$k \approx \frac{1}{l}\sqrt{\frac{m}{M}} \Rightarrow \omega = vk \approx \sqrt{\frac{\tau l}{m}}\frac{1}{l}\sqrt{\frac{m}{M}} = \sqrt{\frac{\tau}{Ml}}.$

Second normal mode:

$k \approx \frac{\pi}{l} \Rightarrow \omega = vk \approx \sqrt{\frac{\tau l}{m}}\frac{\pi}{l} = \pi\sqrt{\frac{\tau}{ml}}.$

(e) For $m << M$, calculate the ratio of the kinetic energy of the string to that of the mass $M$ at the lowest normal mode frequency. (2 marks)

如果 $m << M$，试计算在最低频的正则模中，弦的动能与质量 $M$ 的动能的比例。(2 分)

Let $A$ be the amplitude of oscillation of the mass $M$.

At the first normal mode, the amplitude of oscillation at position $x$ of the string is $Ax/l$.

Hence the ratio of the kinetic energies is $R = \frac{\int\left(\frac{Ax}{l}\right)^2 dm}{MA^2} = \frac{\int x^2 dm}{Ml^2}.$

Note that the numerator is the moment of inertia of a rod about one end.

Hence $R = \frac{ml^2/3}{Ml^2} = \frac{m}{3M}.$

(f) If $m >> M$, find the two lowest normal mode frequency. (2 marks)

如果 $m >> M$, 找出最低頻的兩個正則模頻率。(2 分)

First normal mode:

$kl\tan kl = \frac{m}{M} \gg 1 \Rightarrow kl \approx \frac{\pi}{2} \Rightarrow \omega = vk \approx \sqrt{\frac{\tau l}{m}}\frac{\pi}{2l} = \frac{\pi}{2}\sqrt{\frac{\tau}{ml}}.$

Second normal mode: $kl \approx \frac{3\pi}{2} \Rightarrow \omega = vk \approx \sqrt{\frac{\tau l}{m}}\frac{3\pi}{2l} = \frac{3\pi}{2}\sqrt{\frac{\tau}{ml}}.$

(g) For $m >> M$, calculate the ratio of the kinetic energy of the string to that of the mass $M$ at the lowest normal mode frequency. (2 marks)

如果 $m >> M$，试计算在最低频的正则模中，弦的动能与质量 $M$ 的动能的比例。(2 分)

Let $A$ be the amplitude of oscillation of the mass $M$.

At the first normal mode, the amplitude of oscillation at position $x$ of the string is $A\sin(\pi x/2l)$.

Hence the ratio of the kinetic energies is $R = \frac{\int\left(A\sin\frac{\pi x}{2l}\right)^2 dm}{MA^2} = \frac{\langle\sin^2\frac{\pi x}{2l}\rangle m}{M} = \frac{m}{2M}.$

(h) A traveling wave of angular frequency $\omega$ is generated near the end $x = l$. It propagates towards the mass $M$ and is reflected with a phase shift of $\pi/2$. What is the value of $\omega$ in terms of $\tau$, $m$, $M$ and $l$? (4 marks)

在靠近 $x = l$ 处生成一角频率为$\omega$的行波。该行波朝质量 $M$ 方向传播，并以$\pi/2$ 的相移被反射。 求$\omega$，答案以$\tau$、$m$、$M$ 和 $l$ 表达。(4 分)

The wave can be written as $y(x,t) = A\sin(k(x-l)-\omega t) + rA\sin\left(k(x-l)+\omega t+\frac{\pi}{2}\right)$

$= A\sin(k(x-l)-\omega t) + rA\cos(k(x-l)+\omega t).$

Substituting into the boundary condition,

$M\left.\frac{\partial^2 y}{\partial t^2}\right|_{x=l} = M\omega^2 A[\sin(\omega t) - r\cos(\omega t)],$

$-\tau\left.\frac{\partial y}{\partial x}\right|_{x=l} = \tau kA[-\cos(\omega t) + r\sin(\omega t)].$

For the boundary condition to satisfy for all $t$, we should have $M\omega^2 = \tau kr$ and $M\omega^2 r = \tau k \Rightarrow$ $r = \frac{M\omega^2}{\tau k} = \frac{\tau k}{M\omega^2} \Rightarrow M\omega^2 = \tau k$ and $r = 1$. Hence

$M\omega^2 = \frac{\tau\omega}{v} \Rightarrow \omega = \frac{\tau}{Mv} = \frac{1}{M}\sqrt{\frac{m\tau}{l}}$.

### 3. Maximum Mass of a Star (22 marks) 星体的最大质量 (22 分)

Consider a star of mass *M* and radius *R*. Assume that its density is uniform.
考虑一质量为 *M*、半径为 *R* 的星体。 假设其质量密度均匀。

(a) Its gravitational potential energy *U* can be calculated by considering the work done in bringing a thin layer of materials and depositing on the surface of a spherical protostar of radius *r* when *r* gradually grows from 0 to *R*. Calculate *U*. Express your answer in terms of *G*, *M* and *R*, where *G* is the gravitational constant. (4 marks)
要计算星体的引力势能 *U*，可考虑逐层将星体物质加至半径为 *r* 的球状准星体表面所作的功，并让 *r* 由 0 逐渐增加到 *R*。计算 *U*，答案以引力常数 *G*、*M* 和 *R* 表达。(4 分)

Work done in depositing a layer of thickness $dr$ on the surface of a protostar is the gravitational potential energy change of a layer of volume $4\pi r^2 dr$ brought in from infinity to distance $r$

$dU = -\frac{Gm(r)}{r}\rho 4\pi r^2 dr$, where $\rho = \frac{3M}{4\pi R^3}$ is the density and $m(r) = M\left(\frac{r}{R}\right)^3$ is the mass of the protostar of radius $r$. Hence

$$U = -\int_0^R \frac{GM}{r}\left(\frac{r}{R}\right)^3 \rho 4\pi r^2 dr = -\frac{4\pi\rho GM}{R^3}\int_0^R r^4 dr = -\frac{4\pi\rho GMR^2}{5} = -\frac{3GM^2}{5R}.$$

(b) Assume that the star is made up of protons and electrons, both behaving as ideal gases. It is known that during the formation of the star, half of the loss in gravitational potential energy is converted to thermal energy, while the other half is radiated away. Derive the temperature *T* of the star in terms of *G*, *M*, *R*, $\bar{m}$ and $k_B$, where $\bar{m}$ is the average mass of protons and electrons, and $k_B$ is the Boltzmann constant. (2 marks)
假设星体由质子和电子组成, 其行为皆为理想气体。已知当星体形成时, 其引力势能的耗损一半会转化为热能, 另一半会被辐射掉。试推导星体温度 *T*，答案以 *G*、*M*、*R*、$\bar{m}$和 $k_B$表达。 这里$\bar{m}$为质子与电子的平均质量，$k_B$ 为玻耳兹曼常数。(2 分)

Using the conservation of energy, $\frac{3}{2}Nk_BT = \frac{1}{2}\left(\frac{3GM^2}{5R}\right)$ where $N = \frac{M}{\bar{m}}$ is the number of protons and electrons. $\Rightarrow T = \frac{GM\bar{m}}{5k_BR}$.

(c) Derive the gas pressure $P_g$ of the star in terms of *G*, *M* and *R*. (2 marks)
试推导星体的气体压力 $P_g$，答案以 *G*、*M* 和 *R* 表达。(2 分)

$$P_g = \frac{Nk_BT}{V} = \frac{GM^2}{5R}\frac{3}{4\pi R^3} = \frac{3GM^2}{20\pi R^4}.$$

(d) The virial theorem states that the total pressure in a star is related to the gravitational potential energy by $P = -b\dfrac{U}{V}$. What is the value of $b$? (1 mark)

根据维里定理, 星体内的总压强与引力势能的关系为 $P = -b\dfrac{U}{V}$。求 $b$ 的值。(1 分)

Since $\dfrac{3}{2}Nk_BT = \dfrac{1}{2}\left(\dfrac{3GM^2}{5R}\right)$, we have $\dfrac{3}{2}PV = \dfrac{1}{2}(-U) \Rightarrow b = -\dfrac{PV}{U} = \dfrac{1}{3}$.

(e) At high temperature, photons in the star also contribute to the pressure. Derive the radiation pressure $P_r$ by applying the kinetic theory of gases in a cubic box of volume $L^3$, in which the momenta of the photons are described by the de Broglie relation. Express your result in terms of the photon energy density $u$. (5 marks)

在高温下，星体中的光子也会施加压强。试应用气体运动论，考虑一体积为 $L^3$ 的立方盒子中的光子，其中光子的动量满足德布罗意关系式，从而得出辐射压强 $P_r$。答案以光子能量密度 $u$ 表达。(5 分)

Consider a box of width $L$. Let $p_x$ be the momentum of a photon along the $x$ direction.
Change in momentum of the photon when it hits the wall = $-2p_x$.
Time interval between two successive hits on the wall $= \dfrac{2L}{c\cos\theta_x}$, where $\theta_x$ is the angle between the photon momentum and the $x$ axis.
Using Newton's second law, the force on the wall is the rate of change of momenta of the photons when they hit the wall $F = N\left\langle 2p_x\dfrac{c\cos\theta_x}{2L}\right\rangle = N\left\langle\dfrac{pc\cos^2\theta_x}{L}\right\rangle$, where $N$ is the number of photons, and $\langle\ \rangle$ represents the average over the direction and magnitude of the photon momenta.
Note that for isotropic distributions, $\langle\cos^2\theta_x\rangle = \dfrac{1}{3}$. Pressure: $P_r = \dfrac{F}{L^2} = \dfrac{N}{3L^3}\langle pc\rangle$.
Using de Broglie relation, $pc = \dfrac{hc}{\lambda} = hf = \varepsilon$, where $\varepsilon$ is the photon energy. Hence $N\langle pc\rangle$ is the total photon energy. Hence $P_r = \dfrac{E}{3V} = \dfrac{u}{3}$.

(f) It is known that the photon energy density is given by $u = aT^4$, where $a$ is determined from fundamental constants. Show that $\dfrac{P_r}{P_g} \propto M^c$. What is the value of $c$? (2 marks)

已知光子的能量密度为 $u = aT^4$, 其中 $a$ 由基本常数决定。证明 $\dfrac{P_r}{P_g} \propto M^c$, 并求 $c$ 的值。(2 分)

$$P_r = \frac{a}{3}T^4 = \frac{a}{3}\left(\frac{GM\overline{m}}{5Rk_B}\right)^4 = \frac{aG^4M^4\overline{m}^4}{1875R^4k_B^4}.$$

$$\frac{P_r}{P_g}=\left(\frac{aG^4M^4\overline{m}^4}{1875R^4k_B^4}\right)\left(\frac{20\pi R^4}{3GM^2}\right)=\frac{4\pi aG^3m_H^4}{1125k_B^4}M^2.$$ Hence $\frac{P_r}{P_g}\propto M^2$ and $c = 2$.

(g) Calculate the ratio $\frac{P_r}{P_g}$ for the Sun. You may use the following parameters: (1 mark)

计算太阳的$\frac{P_r}{P_g}$。你可以使用以下参数: (1 分)

$a = 7.565\times10^{-16}$ JK$^{-4}$m$^{-3}$, $G = 6.673\times10^{-11}$ Nm$^2$kg$^{-2}$, $\overline{m} = 8.368\times10^{-28}$ kg, $M_{Sun} = 1.989\times10^{30}$ kg, $k_B = 1.381\times10^{-23}$ JK$^{-1}$.

$$\frac{P_r}{P_g}=\frac{4\pi(7.565\times10^{-16})(6.673\times10^{-11})^3(8.368\times10^{-28})^4}{1125(1.381\times10^{-23})^4}(1.989\times10^{30})^2=1.34\times10^{-4}$$

(h) For stars more massive than the Sun, the radiation pressure becomes increasingly significant and the star becomes unstable. This implies that there is an upper limit on the mass of stable stars. Suppose the radiation pressure becomes equal to 1/3 the gas pressure at this limit. Calculate the temperature in terms of $a$, $k_B$, $\overline{m}$, $M$ and $R$. (2 marks)

对于比太阳重的星体，辐射压强越变得重要，星体也变得越不稳定。这意味着稳定的星体有一个质量上限。假设在这个上限时辐射压强等于 1/3 气体压强。试计算此上限的温度，答案以 $a$、$k_B$,、$\overline{m}$、$M$ 和 $R$.表达。 (2 分)

When $P_r=\frac{1}{3}P_g$, $\frac{a}{3}T^4=\frac{Nk_BT}{3V}\Rightarrow T=\left(\frac{Nk_B}{aV}\right)^{\frac{1}{3}}=\left(\frac{3Mk_B}{4\pi a\overline{m}R^3}\right)^{\frac{1}{3}}$

(i) Using the virial theorem in part (d), find this upper limit of stellar mass. Express your answer in units of solar mass. (3 marks)

应用(d)部中的维里定理, 求星体质量的上限。答案以太阳质量为单位。(3 分)

$$P_g+P_r=-\frac{U}{3V}=-\frac{1}{3V}\left(-\frac{3GM^2}{5R}\right)=\frac{GM^2}{5RV}$$

Since $P_r=\frac{1}{3}P_g$, $\frac{4}{3}P_g=\frac{4Nk_BT}{3V}=\frac{GM^2}{5RV}\Rightarrow T=\frac{3GM^2}{20Nk_BR}=\frac{3GM\overline{m}}{20k_BR}$.

Combining with the result of part (g),

$$\frac{3GM\overline{m}}{20k_BR}=\left(\frac{3Mk_B}{4\pi a\overline{m}R^3}\right)^{\frac{1}{3}}\Rightarrow$$

$$M=\left(\frac{2000k_B^4}{9\pi aG^3\overline{m}^4}\right)^{\frac{1}{2}}=\left[\frac{(2000)(1.381\times10^{-23})^4}{9\pi(7.565\times10^{-16})(6.673\times10^{-11})^3(8.368\times10^{-28})^4}\right]^{\frac{1}{2}}=1.528\times10^{32}\text{ kg}$$

$= 76.8\ M_{Sun}$

Remark: Although this estimate is based on the assumption of uniform stellar density, it agrees with the observation that stars with masses greater than 50 solar masses are rare.

**《THE END 完》**