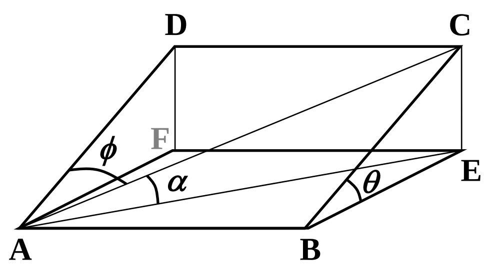
Pan Pearl River Delta Physics Olympiad 2017
2017 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
香港科技大学高等研究院赞助
Simplified Chinese Part-1 (Total 7 Problems, 45 Points) 简体版卷-1（共7题，45分）
(9:00 am – 12:00 pm, 3 February, 2017)

1. No-Shadow Day (5 points) 立竿无影（5 分）

(a) In the figure, ABCD is a rectangle lying on an inclined plane making an angle θ with the horizontal plane. ABEF is the projection of the rectangle on the horizontal plane. If the measure of the angle DAC is ϕ, derive an expression for the angle α. [1]
如图所示，矩形 ABCD 位于斜面上，斜面与水平面夹角为θ。ABEF 为该矩形于水平面的投影。设角 DAC 为ϕ，试推导角α的表达式。[1]

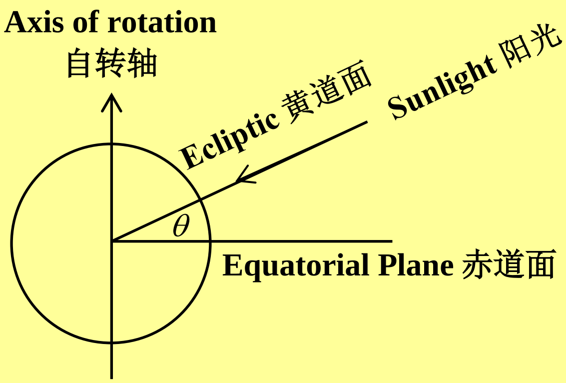


Let h = AC. Then
$\mathrm{CE} = h\sin\alpha$.
$\mathrm{BC} = \mathrm{AD} = h\cos\phi$.
$\mathrm{CE} = \mathrm{BC}\sin\theta = h\cos\phi\sin\theta$.
Equating the expressions of CE, $h\sin\alpha = h\cos\phi\sin\theta \Rightarrow \alpha = \arcsin(\cos\phi\sin\theta)$.

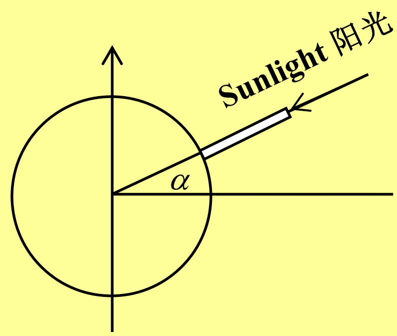
(b) The ecliptic is the plane on which the Earth revolves around the Sun. The axis of rotation of the Earth is inclined at an angle of 23.4° with the normal to the ecliptic. The day of the Summer Solstice (in the Northern Hemisphere) is 21 June. The latitude of Hong Kong is 22.25°, and the no-shadow days are those days on which the Sun does not cast a shadow of a vertical pole at noon in Hong Kong. Using the result of (a) or otherwise, derive the angular displacement of the Earth's revolution between the Summer Solstice and the no-shadow days in Hong Kong. Give your answer to 3 significant figures. [2]
黄道面是指地球围绕太阳公转的平面。地球的自转轴相对于黄道面的法线倾斜，角度为 23.4°。在北半球，夏至的日期为 6 月 21 日。香港位于北纬 22.25°，而当某日正午的太阳照在一支立于香港的垂直竿子时是没有影子的，那日就是香港的无影日了。试用(a)部结果或其他方法，推导在夏至和香港的无影日之间，地球公转的角位移。答案请给三位有效数字。[2]

In the figure above, consider ABEF to be the equatorial plane of the Earth, and ABCD the ecliptic. Then θ = 23.4°. When the Earth revolves around the Sun, sunlight is incident on the Earth from different directions lying on the plane ABCD. For example, on 21 June, sunlight is incident on the Earth in the direction DA, since this is the northernmost direction of sunlight.

Similarly, during Spring Equinox and Autumn Equinox, sunlight is incident on the Earth in the direction AB or BA.



Summer Solstice 夏至



No Shadow Day in Hong Kong
香港无影日

Identifying θ = 23.4^0 and when the Sun does not cast a shadow of the vertical pole at noon in Hong Kong, $\alpha = 22.25^o$.

Hence the angle ϕ is given by

$$\cos\phi = \frac{\sin\alpha}{\sin\theta} = \frac{\sin 22.25^0}{\sin 23.4^0} = 0.9534 \Rightarrow \phi = 17.56^0$$

(c) Write the dates of the no-shadow days in Hong Kong. [2]
试写下香港无影日的日期。[2]

The number of days for the Earth to revolve around the Sun through this angle

$$= 365\left(\frac{17.56}{360}\right) = 17.8$$

Hence the days with no shadow in Hong Kong are 18 days before and after the Summer Solstice, that is, 3 June and 9 July.

2. Six Missiles (5 points) 六枚飞弹 （5 分）

Six missiles are initially located at the six vertex of a regular hexagon with side length *a*. The speed of the missiles in the plane is *v*. Each missile is equipped with an automatic navigation system. The automatic navigation system of each missile guides itself to aim at the current position of its counterclockwise neighbor.

今有飞弹六枚, 分别位于一边长为 *a* 的正六边形的六个角上。 每枚飞弹都装置有自动导航系统。 该系统会指示飞弹永远以速率 v 飞向其逆时针方向之近邻。

(a) Find the radial component of the missile velocities relative to the center of the hexagon. [2]
找出飞弹指向六角形中心的径向速率。[2]

By symmetry, all the six missiles hit at the same time.
By symmetry, they must hit at the center of the hexagon.
By symmetry, the missiles are always at the vertex of a rotating hexagon.
The radial speed is $v\cos(\pi/3) = v/2$.

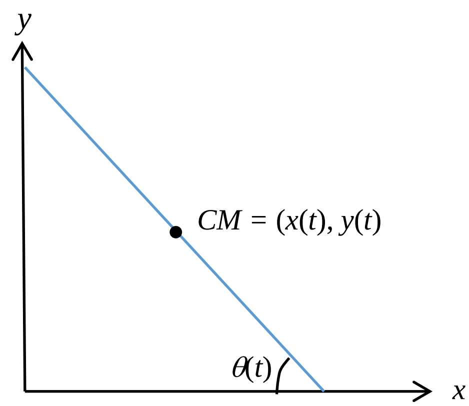
(b) Find the time taken for a missile to hit another. [3]
找出一枚飞弹击中另一枚所需的时间。[3]

The time taken is $a/(v/2) = 2a/v$.

3. Falling ladder (10 points) 下跌中的梯子（10 分）

A ladder of length $2l$ and mass m is standing up against a vertical wall with initial angle α relative to the horizontal. There is no friction between the ladder and the wall or the floor. The ladder begins to slide down with zero initial velocity. Denote $\theta(t)$ as the angle the ladder makes with the horizontal after it starts to slide and $(x(t), y(t))$ be the coordinate of the center of mass of the ladder. In this problem, you should take the gravitational potential energy to be zero at $y = 0$.

一个长度为$2l$、质量为 m 的梯子靠着一道垂直的墙，并与水平形成初始夹角 α。梯子与墙身和地面并没有摩擦力。梯子从零初始速度开始下滑。$\theta(t)$表示为梯子下滑期间与水平形成的夹角, $(x(t), y(t))$ 表示为梯子质心的坐标。在本题中，你应将在 $y = 0$ 处的引力势能取值为零。



(a) What is the initial total mechanical energy of the ladder in terms of α? [1]
梯子的初始总机械能是甚么？答案以α表示。[1]

$E_i = mgy = mgl\sin\alpha$

(b) Write the potential energy of the ladder in terms of $\theta(t)$ when it is sliding. [1]
请用$\theta(t)$写下梯子下滑时的势能。[1]

$U = mgy = mgl\sin\theta$

(c) Write the total kinetic energy of the ladder in terms of $\dot{x}(t), \dot{y}(t), \dot{\theta}(t)$ when it is sliding. (Hint: The moment of inertia of a rod of length $2l$ and mass m about an axis through the center of mass and perpendicular to its length is $I = ml^2/3$.) [1]
请用$\dot{x}(t), \dot{y}(t), \dot{\theta}(t)$写下梯子下滑时的总动能。（提示: 一条长度为$2l$、质量为 m 的杆子，相对於通过杆子质心并垂直于杆子的转动轴，其轉动惯量为$I = ml^2/3$。) [1]

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{6}ml^2\dot{\theta}^2$$

(d) As long as the ladder is in contact with the wall, find the relation between $x(t)$ and $\theta(t)$, and similarly the relation between $y(t)$ and $\theta(t)$. [2]
当梯子靠着墙的时候，请找出$x(t)$和$\theta(t)$的关系式，而同样地，找出$y(t)$和$\theta(t)$的关系式。[2]

$x = l\cos\theta$ and $y = l\sin\theta$

(e) Write the total mechanical energy of the ladder in terms of $\theta(t)$ and $\dot{\theta}(t)$ only by eliminating any dependence on $x(t)$ and $y(t)$. [2]
在消去$x(t)$和 $y(t)$后，只用$\theta(t)$ 和 $\dot{\theta}(t)$写下梯子的总机械能。[2]

$$\dot{x} = -(l\sin\theta)\dot{\theta} \quad \text{and} \quad \dot{y} = (l\cos\theta)\dot{\theta}$$

$$E = T + U = \frac{1}{2}ml^2\dot{\theta}^2(\sin^2\theta + \cos^2\theta) + \frac{1}{6}ml^2\dot{\theta}^2 + mgl\sin\theta = \frac{2}{3}ml^2\dot{\theta}^2 + mgl\sin\theta$$

(f) Derive the relation between $\theta(t)$ and $\ddot{\theta}(t)$. [1]
试推导$\theta(t)$和$\ddot{\theta}(t)$的关系式。[1]

$$\frac{2}{3}ml^2\dot{\theta}^2 + mgl\sin\theta = mgl\sin\alpha \Rightarrow \frac{4}{3}ml^2\dot{\theta}\ddot{\theta} + mgl\cos\theta\dot{\theta} = 0 \Rightarrow \ddot{\theta} = -\frac{3g}{4l}\cos\theta$$

(g) Find the angle θ_c when the ladder loses contact with the vertical wall. [2]
找出当梯子和墙身失去接触时的角度 θ_c。[2]

$$m\ddot{x} = -ml\sin\theta\ddot{\theta} - ml\cos\theta\dot{\theta}^2 = 0 \text{ and } \dot{\theta}^2 = \frac{3g}{2l}(\sin\alpha - \sin\theta)$$

$$\Rightarrow -ml\sin\theta\left(-\frac{3g}{4l}\cos\theta\right) - ml\cos\theta\frac{3g}{2l}(\sin\alpha - \sin\theta) = 0$$

$$\Rightarrow \frac{3}{4}\sin\theta - \frac{3}{2}\sin\alpha + \frac{3}{2}\sin\theta = 0 \Rightarrow \sin\theta_c = \frac{2}{3}\sin\alpha$$

4. Photon Gas (5 points) 光子气体（5 分）

The kinetic theory is very useful in understanding the properties of gases. In this problem we apply the theory to a gas of N photons inside a cubic box with side length L. The energy-momentum relation of a photon is given by $E = |\mathbf{p}|c$.
分子运动理论于理解气体性质时非常有用。 在本题中我们把这理论应用於一长度为 L 的立方盒子中的 N 粒光子。 光子的能量-动量关系为 $E = |\mathbf{p}|c$。

(a) Express the time taken between two consecutive collisions of the same wall of the box normal to the x direction in terms of L, p, p_x and c. [2]
找出一光子连续两次撞击同一面向 x 方向的盒壁之间的时间间距。答案以 L, p, p_x和 c 表达。[2]

During every collision with the wall, the change in momentum is $2p_x$.
The time taken between two collisions is $\Delta t = 2L/c(p_x/p) = 2Lp/cp_x$.

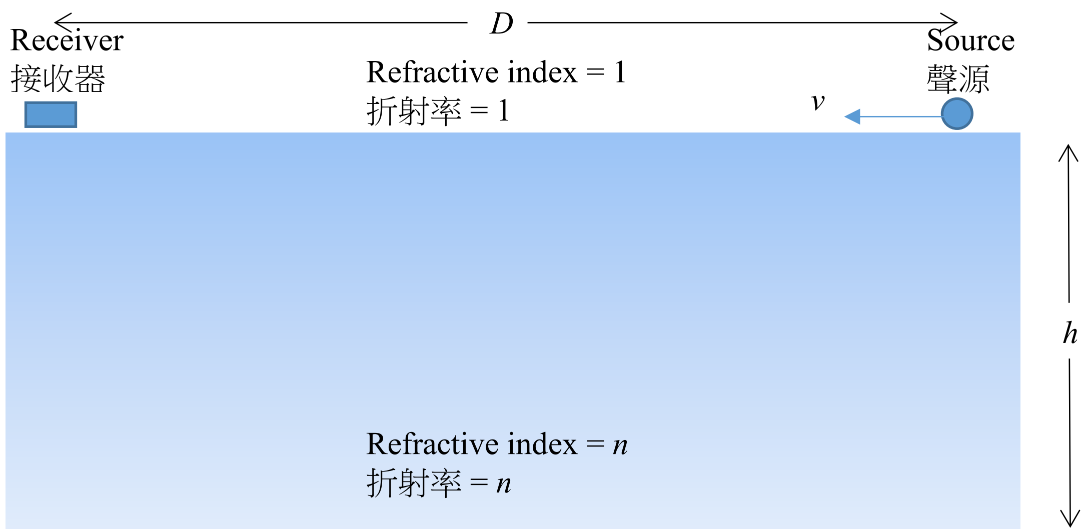
(b) Express the internal energy U of the photon gas in terms of its pressure P and volume V. [3]
找出光子气体的内能 U。 答案以气体的压力 P 和体积 V 表达。[3]

The force is $2p_x/(2Lp/cp_x) = cp_x^2/pL$
The pressure is cp_x^2/pV.
The total pressure is P
$= c\,\langle p_x^2/p\rangle\, N/V = 1/3\, N/V\, c\langle(p_x^2 + p_y^2 + p_z^2)/p\rangle = 1/3\, N/V\, c\langle p\rangle = 1/3\, U/V$
Hence $U = 3PV$.

5. Sea Surface Sound Transmission (5 points) 海面传音（5 分）

In a region where the ocean has a constant depth of h, a sound source emits sound wave with frequency f. Suppose the frequency is so high that sound waves can be treated as rays, and the refractive index of water for sound wave is n. Let the sound speed in air be c. The source is moving with constant horizontal speed $v < c$ towards a stationary receiver at distance D, both located just above the ocean surface, as shown in the figure. It is also assumed that the speed is low so that $\frac{D}{\tau} \gg v$, where $\tau \gg \frac{1}{f}$ is the observation time. Assume we can ignore reflection by the ocean surface and consider only reflection by the ocean floor.

在一处海床深度为常数 h 的海面上，一声源发出频率为 f 的声波。 假设该频率足够高使得声波可以被视为声线束，而海水对声线束的折射率为 n。设空气中的声速为 c。如图中所示，声源与一静止接收器皆位于海面上，两者距离为 D。 声源以均匀速率 $v < c$ 向接收器移动。假设该速率足够慢，使得$\frac{D}{\tau} \gg v$，其中$\tau \gg \frac{1}{f}$为观察时间。 假设海面的反射可以忽略，只需考虑海床的反射。



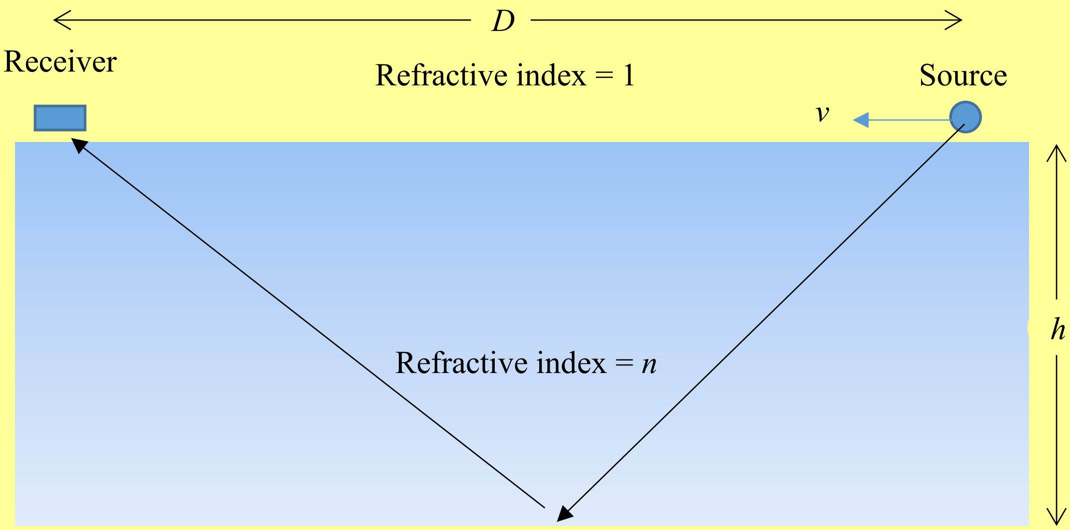
在(a)至(c)中，答案以 c, f, n, 和 v 表达。

(a) Find the frequency of the sound arriving at the receiver through air. [2]
找出声音经过空气到达接收器的频率。[2]

The Doppler shifted frequency is $f_1 = \frac{c}{c-v} f$.

(b) Find the frequency of the sound that will arrive at the receiver via the ocean when $D = 2h$. [2]
找出当$D = 2h$时发出的声音经过海洋到达接收器的频率。[2]

The shortest path via the ocean is reflected at the midpoint:



Component of the source velocity longitudinal to the sound wave in the ocean

$$v' = \frac{(D/2)v}{\sqrt{(\frac{D}{2})^2 + h^2}} = \frac{Dv}{\sqrt{D^2 + 4h^2}}$$

The Doppler shifted frequency is $f_2 = \frac{c/n}{c/n - v'} f = \frac{cf}{c - \frac{nDv}{\sqrt{D^2+4h^2}}}$

When $D = 2h$, we have

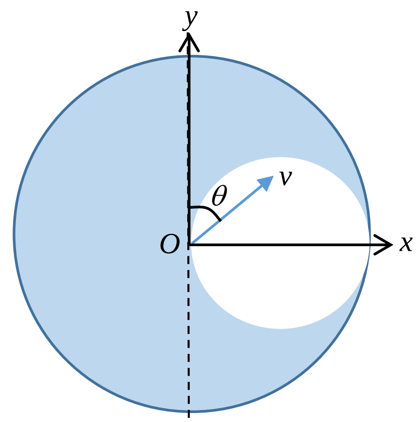
$$f_2 = \frac{cf}{c - \frac{nv}{\sqrt{2}}}$$

(c) Find the instantaneous beat frequency when the sound waves in (b) reach the receiver. [1]
找出(b)中的声波到达接收器时的瞬时拍频频率。[1]

The beat frequency is $f_b = |f_1 - f_2| = \left| \frac{f}{1-\frac{v}{c}} - \frac{f}{1-\frac{nv}{\sqrt{2}c}} \right| = \frac{\left|1-\frac{n}{\sqrt{2}}\right|}{\left(1-\frac{v}{c}\right)\left(1-\frac{nv}{\sqrt{2}c}\right)} \frac{v}{c} f$

6. Electron Trajectory in a Cavity (10 points) 空腔中的电子轨迹（10 分）

A spherical cavity is carved out from a uniformly positively charged sphere with radius *R*. The radius of the cavity is *R*/2, located at a distance *R*/2 from the center of the large sphere, *O*. The total positive charge in the system is *Q*.
在一个带均匀正电、半径为 *R* 的球体内挖出一半径为 *R*/2 的球形空腔。空腔中心与大球中心 *O* 距离为 *R*/2。 系统的总正电荷为 *Q*。



(a) Consider a point in the cavity at distance r and polar angle θ from the origin. Calculate the x and y components of the electric field at the point. [5]
考虑空腔内一点, 其与原点距离为 r, 极角为θ。计算该点电场的 x 分量和 y 分量。[5]

The configuration is equivalent to a fully filled large sphere with charge $8Q/7$ and a small sphere with charge $-Q/7$.
Electric field due to the large sphere

$$E_x = \frac{8Q}{7}\left(\frac{r}{R}\right)^3 \frac{1}{4\pi\varepsilon_0 r^2}\left(\frac{x}{r}\right) = \frac{2Qx}{7\pi\varepsilon_0 R^3},$$

$$E_y = \frac{8Q}{7}\left(\frac{r}{R}\right)^3 \frac{1}{4\pi\varepsilon_0 r^2}\left(\frac{y}{r}\right) = \frac{2Qy}{7\pi\varepsilon_0 R^3},$$

Electric field due to the small sphere

$$E_x = -\frac{Q}{7}\left(\frac{\sqrt{\left(x-\frac{R}{2}\right)^2+y^2}}{\frac{R}{2}}\right)^3 \frac{1}{4\pi\varepsilon_0\left[\left(x-\frac{R}{2}\right)^2+y^2\right]}\left(\frac{x-\frac{R}{2}}{\sqrt{\left(x-\frac{R}{2}\right)^2+y^2}}\right) = -\frac{2Q\left(x-\frac{R}{2}\right)}{7\pi\varepsilon_0 R^3},$$

$$E_y = -\frac{Q}{7}\left(\frac{\sqrt{\left(x-\frac{R}{2}\right)^2+y^2}}{\frac{R}{2}}\right)^3 \frac{1}{4\pi\varepsilon_0\left[\left(x-\frac{R}{2}\right)^2+y^2\right]}\left(\frac{y}{\sqrt{\left(x-\frac{R}{2}\right)^2+y^2}}\right) = -\frac{2Qy}{7\pi\varepsilon_0 R^3},$$

Total electric field

$$E_x = \frac{2Qx}{7\pi\varepsilon_0 R^3} - \frac{2Q\left(x-\frac{R}{2}\right)}{7\pi\varepsilon_0 R^3} = \frac{Q}{7\pi\varepsilon_0 R^2},$$

$$E_y = 0.$$

(b) As shown in the figure, electrons are emitted from O in all directions with speed v and direction θ ranging between 0 and π, but none of them can reach the opposite end of the diameter of the cavity. Gravitational forces are negligible. Find the equation of the envelope of all trajectories of the electrons. [3]
如图所示，现考虑许多电子由 O 以同样速率 v 向 0 至π的各方向θ射出。但其动能不足以到达空腔的直径对点。忽略万有引力。求所有可能的电子轨迹的包络线方程。[3]

The electron is subject to an effective acceleration g to the left with

$$g = \frac{eQ}{7\pi\varepsilon_0 mR^2}$$

The equation of motion of an electron is $x = vt\sin\theta - \frac{1}{2}gt^2,\ \ y = vt\cos\theta$

Eliminating t, the equation of the trajectory is $x = y\tan\theta - \frac{g\sec^2\theta}{2v^2}y^2$
For a given position, the angle θ required to reach the position is given by
$\frac{gy^2}{2v^2}\tan^2\theta - y\tan\theta + x + \frac{gy^2}{2v^2} = 0$
Solution exists if $y^2 \geq 4\left(\frac{gy^2}{2v^2}\right)\left(x + \frac{gy^2}{2v^2}\right)$
Hence the equation of the envelope of safety is $x = \frac{v^2}{2g} - \frac{g}{2v^2}y^2$

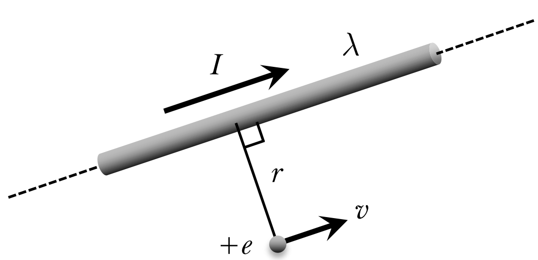
(c) Find the maximum x coordinate where the electrons hit the inner surface of the cavity. Express your answer in terms of Q, the absolute value of the electronic charge e, electron mass m, R, and v. [2]
找出电子打中空腔内表面处最大可能的 x 坐标。答案以 Q、电子电荷绝对值 e、电子质量 m、R 和 v 表达。[2]

The surface of the cavity is given by $(x - \frac{R}{2})^2 + y^2 = R^2/4$
The intersection with the envelope is given by $x = \frac{R}{2} + \frac{v^2}{g} \pm \sqrt{\frac{R^2}{4} + \frac{v^2 R}{g}}$
Since $x = \frac{R}{2} + \frac{v^2}{g} + \sqrt{\frac{R^2}{4} + \frac{v^2 R}{g}} > R$, it is rejected.
So $x = \frac{R}{2} + \frac{v^2}{g} - \sqrt{\frac{R^2}{4} + \frac{v^2 R}{g}}$

7. Proton Motion Near a Charged Current-Carrying Wire (5 points)
带电荷电流线邻近的质子运动（5 分）

As shown in the figure below, a proton of charge $+e$ moves with velocity v parallel to an infinitely-long, thin wire, at a distance r from the axis of the wire. The wire carries a current I, and its charge per unit length is λ (assumed positive and uniform). Both the proton and the wire are in vacuum.
如下图所示，一个带电荷+e 的质子在一条无限长的幼导线附近运动，速度为 v，方向与导线平行。质子与导线轴的距离为 r。导线载着电流 I，且每单位长度电荷为λ(假定为正及均匀)。质子与导线都处于真空中。

Express the answers from (a) to (c) in terms of r, I, λ, the permittivity of vacuum ε_0, the permeability of vacuum μ_0, the speed of light c, and the unit vectors in cylindrical coordinates $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_z$.
在(a)至(c)中，答案以 r、I、λ、真空电容率 ε_0、真空磁导率 μ_0、光速 c 和柱坐标的单位向量 $\mathbf{e}_r$、$\mathbf{e}_\theta$及 $\mathbf{e}_z$ 表示。



(a) Find the electric field experienced by the proton. [2]
求质子所感受到的电场 **E**。[2]

By Gauss's law: $\oint \mathbf{E}\cdot d\mathbf{A} = \dfrac{\lambda}{\varepsilon_0} \Rightarrow \mathbf{E} = \dfrac{\lambda}{2\pi\varepsilon_0 r}\mathbf{e}_r$.

(b) Find the magnetic field experienced by the proton. [2]
求质子所感受到的磁场 **B**。[2]

By Ampere's law: $\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 I \Rightarrow \mathbf{B} = \dfrac{\mu_0 I}{2\pi r}\mathbf{e}_\theta$.

(c) Find the speed of the proton such that it moves in a straight line parallel to the wire. [1]
求质子的速率，使其沿直线平行于导线运动。[1]

Resultant force on the proton is $\mathbf{F}_e + \mathbf{F}_m = e\mathbf{E} + e\mathbf{v}\times\mathbf{B}$

$$= \frac{e\lambda}{2\pi\varepsilon_0 r}\mathbf{e}_r + (ev\mathbf{e}_z)\times\left(\frac{\mu_0 I}{2\pi r}\mathbf{e}_\theta\right) = \frac{e\lambda}{2\pi\varepsilon_0 r}\mathbf{e}_r + \frac{ev\mu_0 I}{2\pi r}(-\mathbf{e}_r) = \left(\frac{e\lambda}{2\pi\varepsilon_0 r} - \frac{ev\mu_0 I}{2\pi r}\right)\mathbf{e}_r$$

For the proton to move in a straight line parallel to the wire:

$$\frac{e\lambda}{2\pi\varepsilon_0 r} = \frac{ev\mu_0 I}{2\pi r} \Rightarrow v = \frac{\lambda}{\varepsilon_0\mu_0 I} = \frac{c^2\lambda}{I}$$

Remark: Some students include relativistic effects in calculating the electric and magnetic fields experienced by the proton. In that case, the correct answer can be obtained by either using (1) $\mathbf{E}' = \gamma(\mathbf{E} + \mathbf{v}\times\mathbf{B})$ and $\mathbf{B}' = \gamma(\mathbf{B} - \mathbf{v}\times\mathbf{E}/c^2)$, or (2) $\lambda' = \gamma(\lambda - vI/c^2)$ and $\mathbf{I}' = \gamma(\mathbf{I} - \mathbf{v}\lambda)$, since ($\rho c$, **J**) is a 4-vector. The answers become $E' = \dfrac{\gamma}{2\pi\varepsilon_0 r}\left(\lambda - \dfrac{v}{c^2}I\right)$ and $B' = \dfrac{\mu_0\gamma}{2\pi r}(I - v\lambda)$.