Physics Enhancement Programme –
Student Self-learning Booklet (try-out version)
物理培訓課程 – 學生自學冊 (試用版)

MECHANICS IN PHYSICS OLYMPIAD
物理奧林匹克力學

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Introduction

This booklet is a self-contained text to cover the topics in mechanics required in Hong Kong Physics Olympiad (HKPhO). It is written in a style that is probably quite different from the normal textbooks, and is intended for those students who wish to explore outside the scope of normal senior secondary school physics curriculum. It is mostly non-calculus based, and only the very elementary calculus is used in a few critical places where it greatly facilitates the understanding of specific subjects. Each subject is presented from its very basics, but it is elevated quite steeply so at the end the level is beyond the normal curriculum. While the key issues are addressed explicitly, details of secondary importance are briefly mentioned and left to the students to study in exercises, and learn physics through the process. Additional references should be sought to further understand the topics like vectors, elementary calculus, etc., most of which can be found in relevant F6-7 textbooks.

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Chapter 1 Vector and Mechanical Motion

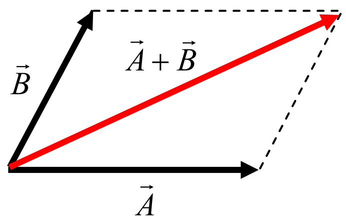
Kinematics means description of motion. In this chapter, we will learn to describe the positions, velocities and accelerations of moving particles. We introduce the use of vector notation and derivative as a rate of change. Readers are assumed to have some basic mathematical background on vectors and calculus.

1.1 Vectors and scalars

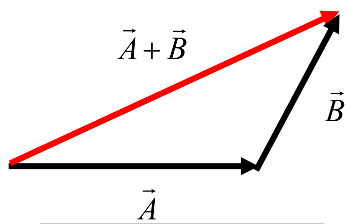
- Vectors are quantities that are specified by a magnitude and a direction.
  e.g. Force, acceleration, velocity, momentum, impulse...
- Scalars are quantities that are specified by a magnitude only.
  e.g. Energy, temperature, charge, mass, speed...

1.2 Addition and subtraction of vectors

- Two vectors are added using either the method of parallelogram or the tip-to-tail method.

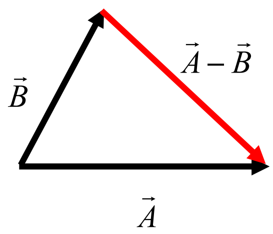


Method of parallelogram

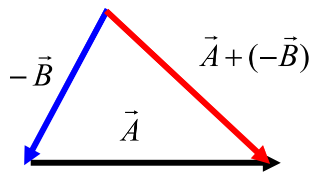


Tip-to-tail method

- $\vec{A}-\vec{B}$ is the vector that add up with $\vec{B}$ to give $\vec{A}$ (see figure on the left). Or we can rewrite $\vec{A}-\vec{B}$ as $\vec{A}+(-\vec{B})$. The negative of the vector $\vec{B}$ is just the vector opposite to it with same magnitude (see figure on the right).



($\vec{A}-\vec{B}$) is the vector that add up with $\vec{B}$ to give $\vec{A}$

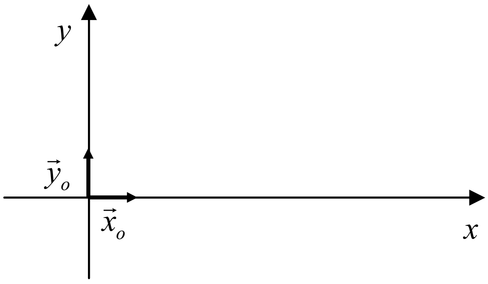


$\vec{A}-\vec{B}=\vec{A}+(-\vec{B})$

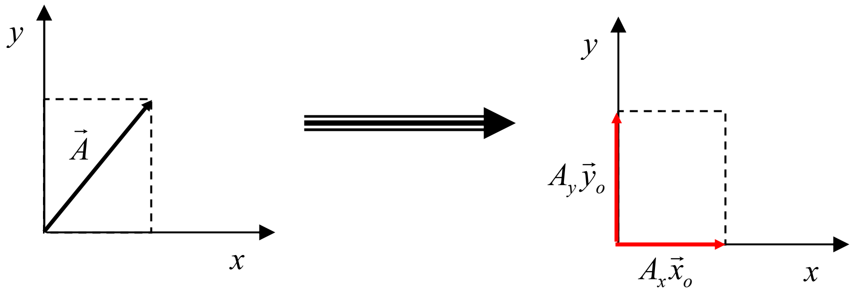
1.3 Resolution of a vector

- A vector can always be *resolved* into *components*. In many cases, by resolving vectors into components, calculations can be simplified.
- To start with, we consider vectors in a 2-dimensional space. Students can easily generalize the results to 3-dimensional cases on their own.
- We define the *unit vector* along the positive *x*-axis be $\vec{x}_o$ and that along the positive

y-axis be $\vec{y}_o$. A *unit vector* is a vector with magnitude 1. Mathematically, we denote the magnitude of a vector $\vec{A}$ by $|\vec{A}|$ (or sometimes simply A). That is, we may write $|\vec{x}_o| = |\vec{y}_o| = 1$.



- In the following diagram, the vector $\vec{A}$ is equivalent to the sum of the two red vectors $A_x\vec{x}_o$ and $A_y\vec{y}_o$. (Addition of vectors) In other words, the vector $\vec{A}$ can be *resolved* into the two red *components*.



- Mathematically, we write

$$\vec{A} = A_x\vec{x}_o + A_y\vec{y}_o \qquad (1.1)$$

where $A_x = A\cos\theta$, θ is the angle that $\vec{A}$ makes with the positive x-axis.
$A_y = A\sin\theta$,

$$A = \sqrt{A_x^{\ 2} + A_y^{\ 2}} \qquad (1.2)$$

- In component form, addition and subtraction can be done "component by component".

If $\vec{A} = A_x\vec{x}_o + A_y\vec{y}_o$ and $\vec{B} = B_x\vec{x}_o + B_y\vec{y}_o$

Then $\vec{A} \pm \vec{B} = (A_x \pm B_x)\vec{x}_o + (A_y \pm B_y)\vec{y}_o \qquad (1.3)$

So adding two vectors can be done by simply adding their corresponding components.

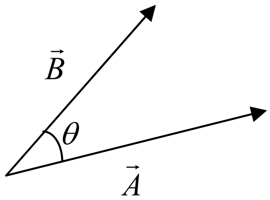
1.4 Dot product and cross product

1.4.1 Dot Product

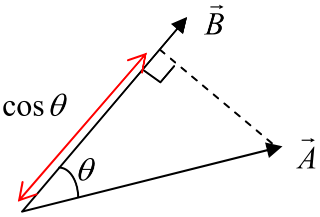
- The dot product (or scalar product) of two vectors $\vec{A}$ and $\vec{B}$ is defined as

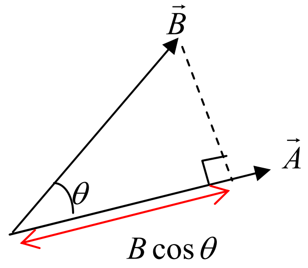
$$\vec{A} \bullet \vec{B} \equiv |\vec{A}| \times |\vec{B}| \times \cos\theta \qquad (1.4),$$

where θ is the angle between the two vectors.



- The dot product is actually the projection of $\vec{A}$ in the direction of $\vec{B}$ times the magnitude of $\vec{B}$. Of course, it is also the projection of $\vec{B}$ in the direction of $\vec{A}$ times the magnitude of $\vec{A}$.





$\vec{A}\bullet\vec{B} = (A\cos\theta)\times B$

$A\cos\theta =$ projection of $\vec{A}$ in the direction of $\vec{B}$

$\vec{A}\bullet\vec{B} = (B\cos\theta)\times A$

$B\cos\theta =$ projection of $\vec{B}$ in the direction of $\vec{A}$

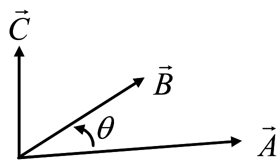
- One example of dot product is the work W done by a force $\vec{F}$ over a distance $\vec{s}$. By definition,

  $W \equiv \vec{F}\bullet\vec{s} = Fs\cos\theta$ (1.5)

- Properties of dot product
  (1) $\vec{A}\bullet\vec{B} = \vec{B}\bullet\vec{A}$ (1.6a)
  (2) $(k\vec{A})\bullet\vec{B} = k(\vec{A}\bullet\vec{B})$, where k is a scalar (1.6b)
  (3) $\vec{A}\bullet(\vec{B}+\vec{C}) = \vec{A}\bullet\vec{B}+\vec{A}\bullet\vec{C}$ (1.6c)
  (4) If $\vec{A}$ and $\vec{B}$ are not parallel to each other,
  $m\vec{A}+n\vec{B}=0$ implies $m = n = 0$ (1.6d)
  (5) For non-zero vectors $\vec{A}$ and $\vec{B}$,
  $\vec{A}\bullet\vec{B}=0$ if and only if $\vec{A}$ is perpendicular to $\vec{B}$ (1.6e)
  (6) If $\vec{A} = A_x\vec{x}_o + A_y\vec{y}_o$ and $\vec{B} = B_x\vec{x}_o + B_y\vec{y}_o$
  Then $\vec{A}\bullet\vec{B} = A_xB_x + A_yB_y$, (1.6f)
  because, $\vec{x}_o\bullet\vec{x}_o = \vec{y}_o\bullet\vec{y}_o = 1$ and $\vec{x}_o\bullet\vec{y}_o = \vec{y}_o\bullet\vec{x}_o = 0$ (1.6g)

1.4.2 Cross Product

The cross product of two vectors $\vec{A}$ and $\vec{B}$ is also a vector $\vec{C}$, denoted by $\vec{C} = \vec{A}\times\vec{B}$. The direction of $\vec{C}$ is perpendicular to the plane formed by $\vec{A}$ and $\vec{B}$, and follows the right-hand rule. Its amplitude is
$C = A\cdot B\cdot\sin\theta$ (1.7).



(Verify that $\vec{B}\times\vec{B} = 0 = \vec{A}\times\vec{A}$)

Note that the order of product is now important. Using the right-hand rule, you can verify that
$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$. (1.8)
(The right-hand convention: use your right hand, extend the thumb, the index finger, and the

middle finger so they are perpendicular to one another. Take the index finger as pointing the direction of $\vec{x}_o$, then the middle finger should be pointing at $\vec{y}_o$ and the thumb pointing at $\vec{z}_o$. Under this definition of axes, $\vec{x}_o \times \vec{y}_o = \vec{z}_o$.)

Cross products also follow the combination rule. That is

$(\vec{A}_1 + \vec{A}_2) \times \vec{B} = \vec{A}_1 \times \vec{B} + \vec{A}_2 \times \vec{B}$; $\vec{B} \times (\vec{A}_1 + \vec{A}_2) = \vec{B} \times \vec{A}_1 + \vec{B} \times \vec{A}_2$ (1.9)

$(k\vec{A}) \times \vec{B} = k(\vec{A} \times \vec{B}) = \vec{A} \times (k\vec{B})$, where k is a scalar. (1.10)

For the unit vectors along the x-y-z axes, we have

$\vec{x}_0 \times \vec{y}_0 = \vec{z}_0$, $\vec{y}_0 \times \vec{z}_0 = \vec{x}_0$, $\vec{z}_0 \times \vec{x}_0 = \vec{y}_0$. (1.11)

Using these results, you can work out the cross product of two vectors in component form. That is $\vec{A} = A_x\vec{x}_0 + A_y\vec{y}_0 + A_z\vec{z}_0$, $\vec{B} = B_x\vec{x}_0 + B_y\vec{y}_0 + B_z\vec{z}_0$, then

$\vec{A} \times \vec{B} = (A_x\vec{x}_0 + A_y\vec{y}_0 + A_z\vec{z}_0) \times (B_x\vec{x}_0 + B_y\vec{y}_0 + B_z\vec{z}_0)$=…..quite long

If you know how to calculate determinant, you can show that

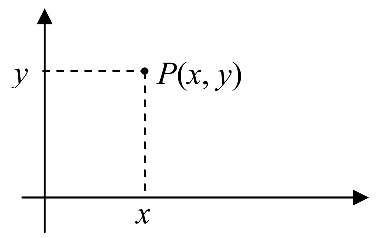
$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{x}_0 & \vec{y}_0 & \vec{z}_0 \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}. \qquad (1.12)$$

It is a handy way to remember.

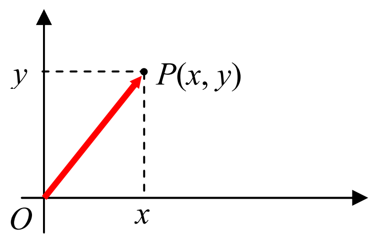
Cross and dot products can be used together, like $(\vec{A} \times \vec{B}) \bullet \vec{C}$, etc.

1.5 Position vectors

- On an x-y plane, we can use a point $P(x, y)$ to represent the position of a particle.

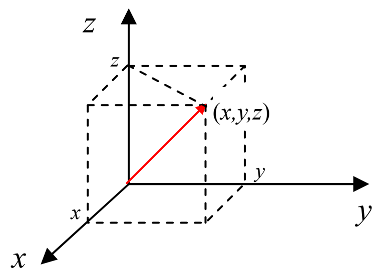


- The position vector $\vec{r}$ of the particle is the vector pointing from the origin O to P.



$$\vec{r} = \overrightarrow{OP} = x\vec{x}_o + y\vec{y}_o \qquad (1.13)$$

- To generalize to 3-dimensional case, we introduce the z-axis which is perpendicular to both x and y-axes and follows the right hand grip rule.



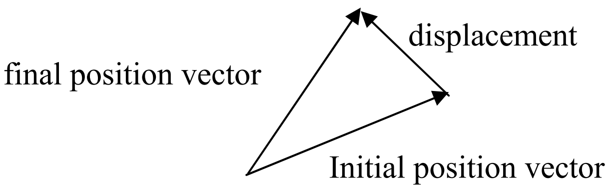
$$\vec{r} = \overrightarrow{OP} = x\vec{x}_o + y\vec{y}_o + z\vec{z}_o \qquad (1.14)$$

You can choose the position of the origin O **anywhere** you want. Usually its position is so chosen that it will not obviously complicate the problem. The directions of $\vec{x}_o$, $\vec{y}_o$ and $\vec{z}_o$ can also be chosen as you wish, as long as they are perpendicular to each other and they form the right-hand convention.

1.6 Displacement, velocity and acceleration

- The change in position vector is called the *displacement* $\Delta\vec{r}$.

  $\Delta\vec{r} =$ final position vector – initial position vector



- In component form, the displacement vector can be written as

  $$\Delta\vec{r} = \Delta x\,\vec{x}_o + \Delta y\,\vec{y}_o + \Delta z\,\vec{z}_o$$

  where Δx, Δy and Δz are the change in x, y and z-coordinates respectively.

- The average velocity vector $\vec{v}_{ave}$ of the particle in a time interval Δt can be written as

  $$\vec{v}_{ave} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\Delta x}{\Delta t}\vec{x}_o + \frac{\Delta y}{\Delta t}\vec{y}_o + \frac{\Delta z}{\Delta t}\vec{z}_o \qquad (1.15)$$

- The instantaneous velocity vector is found by taking the limit that Δt is very small. In order words, the instantaneous velocity is the derivative of position vector with respect to time.

  $$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{x}_0 + \frac{dy}{dt}\vec{y}_0 + \frac{dz}{dt}\vec{z}_0 = v_x\vec{x}_0 + v_y\vec{y}_0 + v_z\vec{z}_0 \qquad (1.16)$$

- The (instantaneous) acceleration is the rate of change of velocity, which is given by

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\vec{x}_0 + \frac{dv_y}{dt}\vec{y}_0 + \frac{dv_z}{dt}\vec{z}_0 = a_x\vec{x}_0 + a_y\vec{y}_0 + a_z\vec{z}_0 \qquad (1.17)$$

  Note that the acceleration is also the second derivative of the position vector.

$$\vec{a} \equiv \frac{d^2\vec{r}}{dt^2} = \frac{d^2x}{dt^2}\vec{x}_0 + \frac{d^2y}{dt^2}\vec{y}_0 + \frac{d^2z}{dt^2}\vec{z}_0 \qquad (1.18)$$

1.7 Derivatives

- In this section, u, v are functions of x and k is a constant.
- The following rules of differentiation are also given without proof.
  - Addition and subtraction rule $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$ (1.19a)
  - Product rule $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$ (1.19b)
  - Quotient rule $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ (1.19c)
  - Chain rule $\frac{d}{dx}f(y) = \frac{df(y)}{dy} \times \frac{dy}{dx}$, (1.19d),

    where $f(y)$ denotes a function of y.

The derivatives of several basic functions are given below without proof.

- $\frac{d}{dx}k = 0$, where k is a constant (1.20a)
- $\frac{d}{dx}ku = k\frac{du}{dx}$ (follows from product rule since $\frac{dk}{dx} = 0$) (1.20b)
- $\frac{d}{dx}x^n = nx^{n-1}$, where n is a constant (1.20c)
- $\frac{d}{dx}\sin x = \cos x$ (1.20d)
- $\frac{d}{dx}\cos x = -\sin x$ (1.20e)

- The derivatives of other functions used in this booklet can be easily done following the above rules. The following are some examples.

  Example: Find $\frac{d}{dt}(3t + \frac{1}{2}kt^2)$, where k is a constant.

  Sol. $\frac{d}{dt}(3t + \frac{1}{2}kt^2) = \frac{d}{dt}(3t) + \frac{d}{dt}(\frac{1}{2}kt^2)$ (addition rule)

  $= 3\frac{dt}{dt} + \frac{1}{2}k\frac{d}{dt}t^2$ (constants can be put in the front)

  $= 3(1) + \frac{1}{2}k(2t^{2-1})$

  $= 3 + kt$

  Example: The displacement of a particle in 1-dimensional motion is given by $s = 3t + \frac{1}{2}kt^2$, find its velocity and acceleration in terms of k and t.

  Sol. From the last example,

$v = \frac{ds}{dt} = 3 + kt$, v increases linearly with time t.

Acceleration is the derivative of v (or the second derivative of x)

$a = \frac{dv}{dt} = \frac{d}{dt}(3 + kt) = k$, a is a constant equal to k.

Example: Find $\frac{d}{dt} r\sin(\omega t)$, where r and ω are constants.

Sol. $\frac{d}{dt} r\sin(\omega t) = r\frac{d}{dt}\sin(\omega t)$ (constants can be put in the front)

$= r\frac{d}{d(\omega t)}\sin(\omega t) \times \frac{d(\omega t)}{dt}$ (chain rule)

$= r\omega\cos(\omega t)$

1.8 Equations of motion for uniform acceleration (projectile motion)

- Recall the equations of motion taught in CE level.

  $v = u + at$ (1.21a)

  $s = ut + \frac{1}{2}at^2$ (1.21b)

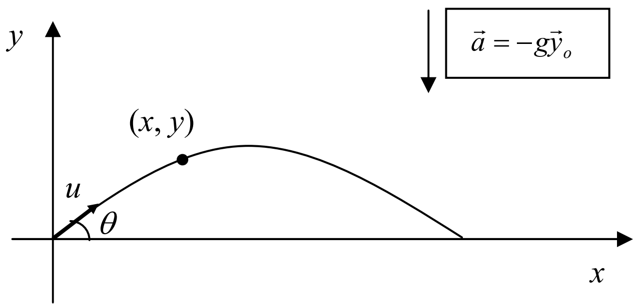
  $v^2 - u^2 = 2as$ (1.21c)

  These three equations can be applied to motions on a straight line with uniform acceleration.

- In reality, objects are moving in a 3-dimensional space. However, if the acceleration of the object is constant, then there must be a certain plane which contains the initial velocity vector and the acceleration vector. (In Geometry two straight lines define a plane. Here the two lines are the acceleration and the initial velocity.) After that, the motion becomes a 2-dimensional problem, because the object has no velocity and acceleration components perpendicular to the plane. In other words, the motion of the object is confined to the plane.

- After the plane is chosen, it is convenient to resolve all vectors into two perpendicular components, one along the acceleration and one perpendicular to the acceleration.

- We consider an object moving under constant gravitational field. We call such motion projectile motion. Choose the initial position of the object as the origin (so $x_0 = y_0 = 0$) and its initial velocity is u with an angle θ to the horizontal ($\vec{x}_o$ direction).



- The motions along the vertical and horizontal directions can be solved independently.
- For horizontal motion,

  $a_x = 0$,

  v_x = constant (i.e. the initial value) = $u_x = u\cos\theta$. (1.22a)

$x - x_0 = u_x\, t$ (1.22b)

- For vertical motion,

  $a_y = -g$ (1.22c)

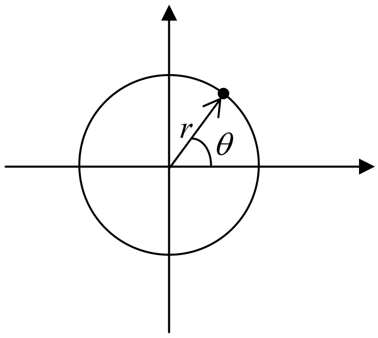
  $v_y = u_y + a_y t = u \sin\theta - gt$ (1.22d)

  $$y - y_0 = u_y t + \frac{1}{2} a_y t^2 = u \sin\theta\ t - \frac{1}{2} g t^2 \quad (1.22e)$$

  (x_0, y_0) is the initial position of the particle at $t = 0$. In this case they are zero. Don't confuse the initial position (x_0, y_0) with the unit vectors $\vec{x}_o$ and $\vec{y}_o$!

- By eliminating t, we can obtain y as a function of x, which is a parabola. It is left to the readers as an exercise.

1.9 Circular motion



- When studying circular motion, it is more convenient to use the polar coordinates (r, θ) to describe the motion of the particle. It is because when the origin is chosen at the centre of the circular path, the magnitude of the position vector r is a constant.

- The position of the particle can be specified by a single parameter θ, which is called the *angular position* of the particle. The unit of θ is radian (rad in short).

- The (instantaneous) *angular velocity* ω is defined as the rate of change of angular position. The unit of ω is rad s^{-1}.

  $$\omega = \frac{d\theta}{dt} \quad (1.23a)$$

- The *angular acceleration* α is defined as the rate of change of angular velocity. Its unit is rad s^{-2}. α is also the second derivative of θ.

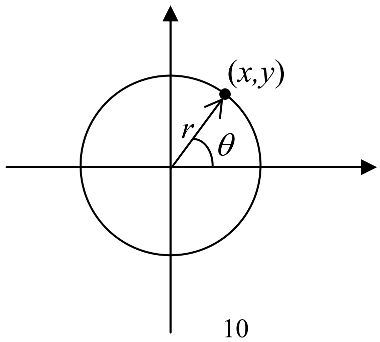
  $$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \quad (1.23b)$$

1.9.1 Uniform circular motion

- For uniform circular motion, ω is constant (or $\alpha = 0$). In this case, θ increases linearly with time.

  $$\theta = \omega t + \phi \quad (1.24)$$

  where ϕ is the initial angular position of the particle. Since it is always possible to choose the axes such that the initial angular position is zero, we let $\phi = 0$ in the rest of this section for simplicity.



- Since the angular velocity is constant, the linear speed v of the particle is also constant. Note that the linear velocity of the particle is NOT constant since its direction is changing.

- If the angular position of a particle is θ, its x and y

coordinates are given by the following equations.

$x = r\cos\theta = r\cos(\omega t)$ (1.25a)

$y = r\sin\theta = r\sin(\omega t)$ (1.25b)

- The x-component and y-component of the velocity $\vec{v}$ of the particle can be found be differentiating the above equations.

$$v_x = \frac{dx}{dt} = r(-\sin(\omega t)) \times \omega = -r\omega\sin(\omega t) \quad (1.26a)$$

$$v_y = \frac{dy}{dt} = r\cos(\omega t) \times \omega = r\omega\cos(\omega t) \quad (1.26b)$$

The magnitude of $\vec{v}$ (i.e. the speed) is given by

$$v = \sqrt{{v_x}^2 + {v_y}^2} = r\omega, \quad (1.27)$$

which is a constant, as expected.

- It can be easily checked that $\vec{r} \bullet \vec{v} = 0$. That is, the position vector is always perpendicular to the velocity vector.

- The acceleration $\vec{a}$ can be found by differentiating $\vec{v}$ with respect to t.

$$a_x = \frac{dv_x}{dt} = -r\omega^2\cos(\omega t) \quad (1.28a)$$

$$a_y = \frac{dv_y}{dt} = -r\omega^2\sin(\omega t) \quad (1.28b)$$

- Since $a_x = -r_x\omega^2$ and $a_y = -r_y\omega^2$

$$\vec{a} = -\omega^2\vec{r} \quad (1.28c)$$

The acceleration vector is opposite to the position vector. That is, the acceleration is always pointing towards the centre of the circular path. This is called the *centripetal acceleration*.

1.9.2 Non-uniform circular motion (Optional. You may skip this part.)

- If ω is not constant ($\alpha \neq 0$), the acceleration consists of two parts.
- The angular position is not linearly related with t any more. We go back to the origin equations.

$x = r\cos\theta$

$y = r\sin\theta$

$$v_x = r(-\sin\theta)\frac{d\theta}{dt} = -r\omega\sin\theta$$

$$v_y = r\cos\theta\frac{d\theta}{dt} = r\omega\cos\theta$$

$$a_x = -r\frac{d\omega}{dt}\sin\theta - r\omega\cos\theta\frac{d\theta}{dt} = -r\alpha\sin\theta - r\omega^2\cos\theta$$

$$a_y = r\frac{d\omega}{dt}\cos\theta + r\omega(-\sin\theta)\frac{d\theta}{dt} = r\alpha\cos\theta - r\omega^2\sin\theta$$

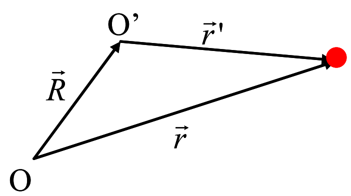
Combining the results, we have

$\vec{a} = r\alpha\hat{e}_\theta - r\omega^2\hat{e}_r = a_T\hat{e}_\theta + a_C\hat{e}_r$

where $\hat{e}_r$ is the unit vector along the radial direction and $\hat{e}_\theta$ is the unit vector is along the tangential direction with increasing θ.

- The second term is just the centripetal acceleration, which is the same as that introduced in uniform circular motion.
  $a_C = -r\omega^2$
- The first term is called the *tangential acceleration*, which is along the tangent of the circular path.
  $a_T = r\alpha$
- Actually, a_T is the rate of change of the linear speed v.
  $a_T = \dfrac{dv}{dt}$
  We can check that if v is a constant, the motion returns to uniform circular motion.

1.10 Relative Motions



A reference frame is needed to describe any motion of an object. Consider two such reference frames S and S' with their origins at O and O', respectively. The X–Y-Z axes in S are parallel to the X'-Y'-Z' axes in S'. One is moving relative the other with a velocity $\vec{u}$ and acceleration $\vec{A}$.

Note: $\vec{r} = \vec{r}\,' + \vec{R}$. (2.1)

So the velocity is: $\vec{v} = \dfrac{d\vec{r}}{dt} = \dfrac{d\vec{r}\,'}{dt} + \dfrac{d\vec{R}}{dt} = \vec{v}\,' + \vec{u}$ (1.29)

Similar for acceleration: $\vec{a} \equiv \dfrac{d\vec{v}}{dt} = \dfrac{d\vec{v}\,'}{dt} + \dfrac{d\vec{u}}{dt} = \vec{a}\,' + \vec{A}$ (1.30)

This is the classic theory of relativity. If $\vec{u}$ is constant, then $\vec{A} = 0$, and $\vec{a} = \vec{a}\,'$, i. e., Newton's Laws work in all inertia reference frames.

Properly choosing a reference frame can sometimes greatly simplify the problems.

1.11 Questions for discussion:

(Brief solutions are provided for question 4, question 8 and question 11. You may take them as examples for the topics presented above.)

1. What is the physical significance of the +/– sign before a vector? Does the +/– sign affect the magnitude of a vector?
2. Can a scalar quantity be negative? If so, what is the physical significance of the +/– sign when compared to that for vector quantities? Give examples.
3. Generalize the results for 2-dimensional vectors to 3-dimensional case. Let $\vec{x}_o$, $\vec{y}_o$ and $\vec{z}_o$ be the unit vectors along the positive x, y and z-axes.

Let $\vec{A} = A_x\vec{x}_o + A_y\vec{y}_o + A_z\vec{z}_o$ and $\vec{B} = B_x\vec{x}_o + B_y\vec{y}_o + B_z\vec{z}_o$.

a. Write down $\vec{A} \pm \vec{B}$.

b. Write down $\vec{A} \bullet \vec{B}$.

c. Write down $\left|\vec{A}\right|$ and $\left|\vec{A}+\vec{B}\right|$.

d. Show that $\vec{A} \bullet \vec{A} = \left|\vec{A}\right|^2$.

e. Determine which of the following are vector and which are scalar.
$(\vec{A}\times\vec{B})\bullet\vec{C}$; $\vec{A}\times(\vec{B}\bullet\vec{C})$; $\vec{A}\times(\vec{B}\times\vec{C})$; $(\vec{A}\times\vec{B})\times\vec{C}$.
Also, examine if the last two are equal, then determine whether $\vec{A}\times\vec{B}\times\vec{C}$ is meaningful.

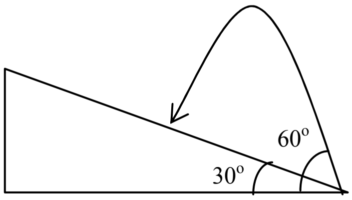
4. A helicopter is trying to land on a ship deck which is drifting south (unit vector $\vec{y}_0$) at 17m/s. A 12m/s wind is blowing from east (unit vector $\vec{x}_0$). The ship crew sees the helicopter descending at 5 m/s. Take the downwards direction as unit vector $\vec{z}_0$. What is its velocity relative to water and air?
   (a) $(5\vec{y}_0 - 17\vec{z}_0)$ m/s; $(-12\vec{x}_0 + 17\vec{y}_0 + 5\vec{z}_0)$ m/s
   (b) $(-12\vec{x}_0 + 17\vec{y}_0 + 5\vec{z}_0)$ m/s; $(17\vec{y}_0 + 5\vec{z}_0)$ m/s
   (c) $(5\vec{z}_0)$ m/s; $(-12\vec{x}_0 + 17\vec{y}_0 + 5\vec{z}_0)$ m/s
   (d) $(17\vec{y}_0 + 5\vec{z}_0)$ m/s; $(-12\vec{x}_0 + 5\vec{z}_0)$ m/s
   (e) $17\vec{y}_0$ m/s; $(-12\vec{x}_0 + 5\vec{z}_0)$ m/s

5. The initial position vector of P is $\vec{r}_0 = 3\vec{x}_o - 4\vec{y}_o + 2\vec{z}_o$ and it accelerates from rest at a constant acceleration $\vec{a} = 2\vec{x}_o - 3\vec{y}_o - \vec{z}_o$. Find its position vector and velocity vector $\vec{r}$ as a function of time *t*. (Answer: $\vec{r} = \vec{r}_0 + \frac{1}{2}\vec{a}t^2$)

6. The position of a particle moving on a straight line is given by $x = A\sin(\omega t)$, where A and ω are constant. (You may call A as the *amplitude* and ω as the *angular frequency* of the motion.) Find the velocity and acceleration with respect to time *t*. Sketch the graphs of *x-t*, *v-t* and *a-t*. What can you say about the relationship between *a* and *x*?

7. An airplane flies horizontally with a constant speed v_o at height *h* above the ground. A cannon launches a shell to hit the airplane when the airplane is exactly above the cannon.
   a. Find the minimum initial speed of the shell if it can hit the airplane.
   b. For the minimum launching speed,
      i. at what angle should the shell be fired?
      ii. find the position and time elapsed when the airplane is hit.

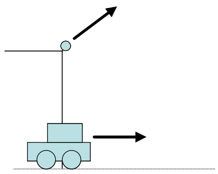
8. An object is projected, with a speed 20 m s^{-1} and an angle 60^o to the horizontal, at the bottom of a slope with inclination angle 30^o. Find the position at which the object hits the slope.



9. A particle is put on a circular table, which is rotating about its central axis with angular speed ω. The particle is 10 cm from the centre. If ω increases gradually from zero and the coefficient of friction between the particle and the table is 0.5. Find the value of ω when the particle starts to slide on the table. (Do it after studying Ch. 3.)

10. A particle is moving along a vertical circular track of radius *r,* under the action of gravity. Its speed at the bottom of the track is *v*. Find the minimum value of *v* such that the particle can complete the circular motion. Discuss the motion of the particle if *v* is smaller than this minimum value.

11. A ball is thrown out at height *H* with initial speed *v* to hit a car moving at speed v_2 when the car is just passing below, find the angle θ necessary for the ball to hit the car, and the position of the car when it is hit. (This quest ion is similar to Q7.)



Further discussions
(1) Can $v_2 > v$?
(2) Can θ be negative? Compare the answer to the case when θ is positive.
(3) Use the car as the reference frame, solve the problem.

1.12 Suggested solution for question:

4.

The boat is moving with the water at $\vec{v}_1 = 17\vec{y}_0$ m/s; the air velocity is $\vec{v}_2 = 12\vec{x}_0$ m/s, all relative to Earth. The velocity of the helicopter in the boat reference frame is $\vec{v}_3 = 5\vec{z}_0$ m/s. Since the boat is moving with water, the relative velocity of the helicopter to water is also $\vec{v}_3 = 5\vec{z}_0$ m/s. Its velocity to Earth is $\vec{v}_4 = \vec{v}_1 + \vec{v}_3$, its velocity to air is then $\vec{v}_5 = \vec{v}_4 - \vec{v}_2$. So answer-c is the correct one.

8.

Let β = 30°, α = 60°

Method-1

Choose x-axis in the horizontal pointing to the *left*, and y-axis pointing upwards.

$x = vt\cos\alpha$, (1)

$y = vt\sin\alpha - \frac{1}{2}gt^2$. (2)

Hitting the slope means $y = x\tan\beta$. (3)

We have three unknowns *x, y, t* and three equations. Eliminate *x, y* in Eq. (3) by Eqs. (1) and (2) to solve for *t*, then put back to get *x* and *y*. You can plug in the numbers yourself.

Method-2

Choose the x-axis along the slope and the y-axis perpendicular to the slope. In such system, $\vec{g} = -g\sin\beta\vec{x}_0 - g\cos\beta\vec{y}_0$, i. e., the motions in both the x-direction and the y-direction are uniform acceleration motion.

$x = vt\cos(\alpha - \beta) - \frac{1}{2}gt^2\sin\beta$, (1)

$y = vt\sin(\alpha - \beta) - \frac{1}{2}gt^2\cos\beta$ (2).

Hitting the slope means *y* = 0. Put it into the equation for *y* we get $t = \frac{2v\sin(\alpha-\beta)}{g\cos\beta}$.

Put it to Eq. (1) for *x* and we are done.

Verify that the *x* here equals to $\sqrt{x^2 + y^2}$ in Method-1.

11.

The initial position of the ball is (0, *H*). Ball: $y_1 = H + vt\sin\theta - \frac{1}{2}gt^2$, $x_1 = vt\cos\theta$

The initial position of the car is (0, 0). Car: $y_2 = 0$, $x_2 = v_2 t$

Ball hits the car means $y_1 = y_2$, and $x_1 = x_2$. This leads to $v_2 = v\cos\theta$, and the angle θ is determined.

Put θ back to $0 = H + vt\sin\theta - \frac{1}{2}gt^2$ to get *t*, then put *t* to one of the two equations for x_1 or x_2 for the horizontal distance.

Chapter 2 Force and Equilibrium of Body

2.1 Force

Basic concept and theory

1. Concept of force

- Force is an interaction between objects and it cannot exist independently without the presence of objects.
- A force can deform an object and/or change its state of motion.
- Forces are vectors. Magnitude, direction and point of action are three important elements of a force.
- The S.I. unit of force is Newton (N).

2. Classification of force

(1) According to its nature : can be classified into gravitational force, elastic force, frictional force, molecular force, magnetic force and nuclear force, etc..

(2) According to its effect : can be classified into pressing force, tension, frictional force, centripetal force, restoring force, etc..

(3) According to its way of action : field force (e.g. gravitation, magnetic force…) and contact force (e.g. elastic force, frictional force, etc.)

(4) Besides gravity, all other forces listed above are the manifestation of electromagnetic force.

3. Common forces in mechanics

(1)Weight

(a) Definition : every objects on earth are acted by gravity. Weight is the force which act on the object due to gravity.

(b) Magnitude : $G = mg$, where m is the mass of the object and g is the gravitational field strength on Earth surface

(c) Direction : downwards

(d) Cener of gravity : Weight is acting on the whole body of a finite object, in which there is a special point. The effect of the weight is the same as if the weight is acting on that particular point of the object.

(2)Elastic force

(a) Definition: Two objects are in contact and distorted by each other. The force to restore their original shape acting on each other is called elastic force.

(b) Direction : Press and tension are perpendicular to the contact surface and towards the object which is being pressed or extended. The direction of tension in a string is along the direction of elongation.

(c) Magnitude : Spring within its elastic limit obeys Hooke's' law $F = kx$. Other kind of elastic force could be calculated by Newton's law or equilibrium condition according to their state of motion.

(3) Friction

(a) Definition: Friction is the force that opposes the relative motion or tendency of such motion of two surfaces in contact.

(b) Static friction : It changes according to the external force or the change of state of motion of an object. Usually, it is not calculated by formula. Instead, the magnitude could be calculated according to the state of motion of an object by using equilibrium condition or Newton's law of motion. The maximum friction force is $F = \mu_s F_N$, where μ_s is the coefficient of static friction and F_N is the normal force to the contact surface.

(c) Kinetic friction : The magnitude can be calculated from the equation $F = \mu_k F_N$, where F is the maximum possible force exerted by friction; μ_k is the coefficient of kinetic friction and F_N is the normal force to the contact surface. The friction force is exerted in the direction along the contact surface of the object and opposite to the relative motion.

(d) Gravitational force

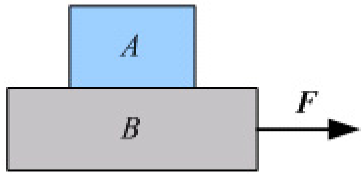
Formula : It exists between any two objects in nature. Its magnitude depends on the product of their masses.

formula： $F = G\dfrac{m_1 m_2}{r^2}$ ， (2.1)

universal gravitational constant $G = 6.67\times10^{11}$ N m^{12}/kg^2 。

Points to note :

1. Weight is due to the gravitational force acting on the objects on the surface of the Earth. This gravitational force can be resolved into weight and the necessary centripetal force to perform circular motion due to the rotation of the Earth.
2. Elastic force exists when the objects are in contact and distorted in shape. The elastic force would not be existed if there is no distortion of shape. Normally, the distortion itself is too small to be observed. The existence of elastic force should be analyzed by its state of motion, conditions of equilibrium and Newton's' laws.
3. Static friction is passive. Its magnitude and direction are determined by external force and state of motion.
4. Friction is the opposition of relative motion and tendency of motion. It is not the opposition of motion of the object. It may be the force that causes the motion. As shown in the figure below, object B is pulled by a force F on a smooth surface. Object A is moved with B by the friction towards the right.



5. When using the universal law of gravitational, the two objects can be seen as point masses. The distance between two uniform spheres or shells could be measured from their centres.

Examples

Example 1:

Determine the correctness of the following statements :

A. There will be elastic force when two objects are in contact.

B. The centre of gravity may not be in the body.

C. Kinetic friction is always opposite to the direction of motion of the object.

D. An object of weight 2N is held by a light spring which is hung on the ceiling. The spring is extended by 2 cm. If the extension is 10cm, then 10N of pulling forces are acted on both ends.

Analysis:

A. Elastic force exists when the two objects are in contact and distorted. So, A is incorrect.

B. Centre of gravity could be outside of a body. The centre of gravity of a semi-circular iron ring can be determined by suspending of a plumb line which is not on the ring. So, B is correct.

C. A small object with initial velocity v_o slips on a wooden board which is on a smooth surface. The wooden board is moved by the kinetic friction with the small object. According to the wooden board, the movement and the kinetic friction are in the same direction. So, C is incorrect.

D. Within the elastic limit, the elastic force is directly proportional to the extension. So, D is correct.

2.2 Addition and resolution of forces

Basic concept and theory

1. Addition of forces – finding resultant force of a number of known forces

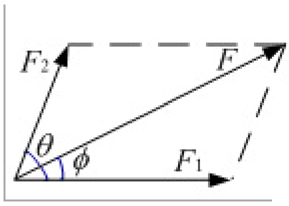
(1) Addition of forces on a straight line which are acting on the same point : Declare a direction as positive. The forces which are in the same direction are taken as positive and those in the opposite direction are taken as negative. The resultant force is the algebraic sum of all the forces. The range of resultant force is $|F_1 - F_2| \le F \le |F_1 + F_2|$.

(2) Addition of forces with an angle

(i) Method of parallelogram

$$F=\sqrt{F_1^{\,2}+F_2^{\,2}+2F_1F_2\cos\theta}$$

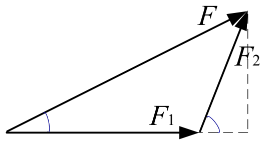
$$\phi=\arctan\frac{F_1\sin\theta}{F_2+F_1\cos\theta}$$



(ii) Method of triangle

$$F=\sqrt{F_1^{\,2}+F_2^{\,2}+2F_1F_2\cos\theta}$$

$$\frac{\sin\phi}{F_2}=\frac{\sin\theta}{F}$$



(iii) Resolving the force into components

$$F=\sqrt{F_x^{\,2}+F_y^{\,2}}\ ,$$

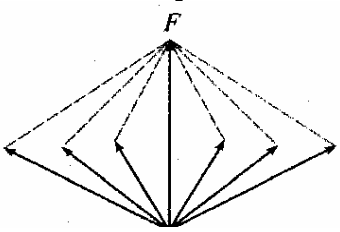
$$\phi=\arctan\frac{F_y}{F_x}$$

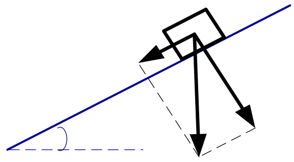
$$F_x=\sum F_{ix}\cos\theta_i,$$

$$F_y=\sum F_{iy}\cos\theta_i$$

2. Resolving the forces

Resolving a force is to find the components of a force. There are infinite combinations of components, as shown in the diagram below.





In general, the way of resolution of force into components is done according to the effect of it. For example, the weight acting an object which is placed on an inclined plane causes the object to slide down the slope and press on the plane. That is, the force could be resolved into two components which are along and normal to the surface.

$$F_1=mg\sin\alpha\text{，}F_2=mg\cos\alpha\text{。}$$

ϕ

Readers may refer to Ch.1 for more about the vectors.

<u>Examples</u>

<u>Example 2</u>

When a force is resolved into two components, which of the following statement is correct ?

A. The resultant force must be equal to the sum of the components.

B. The resultant force must be larger than the sum of the components.

C. The resultant force must be smaller than the sum of the components.

D. The resultant force must be larger than one the component and smaller than the other one.

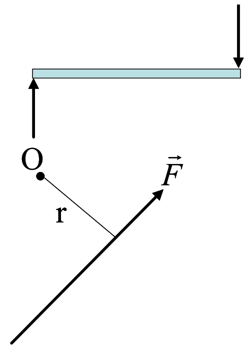
E. The resultant force could be larger or smaller than both of the components. Also, it could be larger or smaller than any one of the components.

<u>Analysis:</u>

The resultant force of two forces satisfy $F_1 - F_2 \le F_{\text{resultant}} \le F_1 + F_2$. So, E is correct.

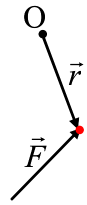
2.3 Torque

When two forces of equal amplitude and opposite directions acting upon the two ends of a rod, the center of the rod remains stationary but the rod will spin around the center. The torque (of a force) is introduced to describe its effect on the rotational motion of the object upon which the force is acting. First, an origin (pivot) point O should be chosen. The amplitude of the torque of force $\vec{F}$ is



$$M = rF \qquad (2.2),$$

where r is the distance between $\vec{F}$ and the origin O. The direction of the torque (a vector as well) is point out of the paper surface using the right hand rule. *One can choose any point as origin*, even if the point is NOT on the object the force is acting upon, so the torque of a force depends on the choice of origin. However, for two forces of equal amplitude and opposite directions, the total torque is independent of the origin. Try it on the example shown in the figure.



The general form of torque is defined as

$$\vec{M} \equiv \vec{r} \times \vec{F} \qquad (2.3),$$

which involves the cross product of two vectors. One can show, using the rules for cross product in Chapter-1, that the amplitude of the torque given by Eq. (2.3) is the same as that in Eq. (2.2).

2.4 Equilibrium of a body

<u>Basic concept and theory</u>

1. State of equilibrium

If an object is at rest or in uniform motion, this object is said to be at equilibrium.

2. Condition of equilibrium

(1) The vector sum of all the external forces acting on the body must be zero.

$$\text{i.e. } \sum \vec{F}_i = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots \vec{F}_n = 0 \qquad (2.4a)$$

(2) The vector sum of all the external torques (moment) acting on the body must be zero.

$$\text{i.e. } \sum \vec{M}_i = \vec{M}_1 + \vec{M}_2 + \vec{M}_3 + \ldots \vec{M}_n = 0 \qquad (2.4b)$$

3. Steps of applying conditions of equilibrium for solving problems

(1) Steps of applying conditions of equilibrium for solving problems of forces acting on a common point

a. consider a specific object and find out all the external forces acting on it.

b. choose a suitable way of addition and resolving the forces.

c. establish equations according to the conditions of equilibrium.

d. solve the simultaneous equations.

(2) Steps of applying conditions of equilibrium to a body with rotational axes

a. recognize the position of the rotational axis.

b. recognize the points of action and their directions.

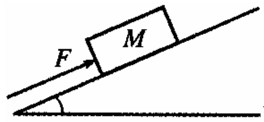
c. determine the moment of forces.

d. establish equations according to the conditions of equilibrium according to the result of the moment of forces, then solve the equations.

Examples

Example 3

As shown in the figure, a force *F* pushes on a block M which is placed on an inclined plane. The block is at rest. Consider the friction acting on the block by the inclined plane, determine whether each of the following statements is correct or not.



A. The friction may be upward along the plane

B. The friction may be downward along the plane

C. The friction may be zero.

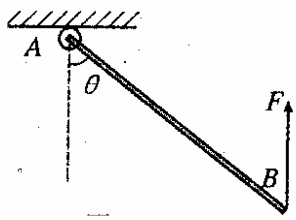
D. The friction may be larger than *F*.

Analysis

The direction of static friction is determined by the tendency of relative motion. The tendency of relative motion is determined by the magnitudes of the component of its weight along the surface of the plane *G* and the force *F*. If $G > F$, then the block *M* tends to slide down and friction is upward. If $G < F$, the block *M* tends to move upward and friction is downward. If $G = F$, then friction is zero. As a result, all of the above are correct.

Example 4

As shown in the figure, a uniform rod can be rotated freely about A on the vertical plane. When a vertical upward force *F* lifts up the rod slowly, which of the following statements is correct ?



A. The pulling force is unchanged, its moment of force increases.

B. The pulling force is unchanged, its moment of force decreases.

C. The pulling force increases, its moment of force decreases.

D. The pulling force decreases, its moment of force decreases.

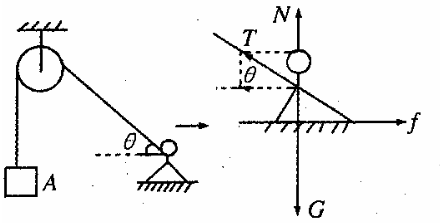
Analysis

The rod is lifted slowly and the rod is kept in equilibrium. The moment *F* should be equal to the moment of the weight of the rod. When θ increases, the moment of weight increases. So, the moment of *F* increases. As the ratio of the moment of weight to *F* is unchanged (1 : 2), *F* is unchanged. A is correct.

2.5 Comprehensive examples

Example 1

As shown in the figure, the man moves to the right and block *A* is lifted up slowly. The normal reaction and friction acting on the man by the ground are *N* and *f* respectively. The tension of the string is *T*. During the movement, which of the following is correct ?



A. *N, f* and *T* increase

B. *N* and *f* increase, *T* is unchanged

C. *N*, *f*, and *T* decrease

D. *N* increases, *f* decreases and *T* is unchanged

Analysis

Tension (*T*=*mg*, where *m* is the mass) in the string is unchanged. Although the man is moving slowly, the friction is static friction.

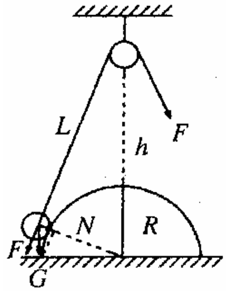
As shown in the diagram, the forces acting on the man are the pulling force of the tension, weight, normal reaction and static friction by the ground.

Resolving *T* into a vertical component $T\sin\theta$ and a horizontal component $T\cos\theta$, the equation is then $N+T\sin\theta=G$. Thus, $N=G-T\sin\theta$. When the man moves to the right, θ becomes smaller and *N* becomes larger.

According to $f=T\cos\theta$, when $\cos\theta$ becomes larger, f becomes larger. The answer is B.

Example 2

As shown in the diagram, a pulley is hung at a distance *h* vertically above a smooth hemisphere with a radius *R*. A small sphere on the hemisphere with weight *G* is attached to a string which is passed through the pulley and pulled slowly by a man on the other side. The small sphere is moved from the bottom to the position which is near the top. What are the changes of the force acting on the surface of the hemisphere by the small sphere and the tension in the string ? (Neglect the radii of the pulley and the small sphere.)



Analysis

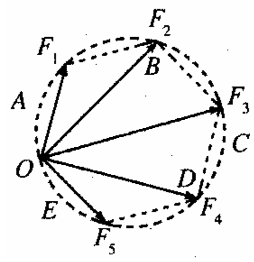
Consider the small sphere, the forces acting on it are the weight, tension and normal reaction. The weight is resolved into F and N which are along the string and perpendicular to the surface respectively.

The triangle G, F, N and L, R, (R+h) are similar. Consider the corresponding sides, we have $\frac{N}{G}=\frac{R}{R+h}$ and $\frac{F}{G}=\frac{L}{R+h}$. During the movement, R, h are unchanged and L is decreased. Then, the force acting on the hemisphere $N=\frac{RG}{R+h}$ is unchanged. Then tension $F=\frac{LG}{R+h}$ is decreased.

We have used some mathematical knowledge (similar triangle) to solve this problem clearly. It is common to transform a problem in physics to a problem in mathematics.

Example 3

As shown in the diagram, the five forces F_1, F_2, F_3, F_4, F_5 are acting on the point O. They construct a hexagon as shown in the diagram. If F_3 = 10N, then find the resultant force.



Analysis

In solving this problem, some properties of parallelogram and hexagon are used.

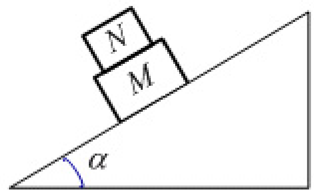
$OACD$ is a parallelogram. F_1 and F_4 are adjacent sides and F_3 is the diagonal. The resultant of F_1 and F_4 is F_3 according to the method of parallelogram. Similarly, consider the parallelogram $OBCE$, the resultant of F_2 and F_5 is F_3. Thus, the resultant of F_1, F_2, F_3, F_4, F_5 is

F = $3F_1$ = 30N.

Example 4

As shown in the diagram, the blocks M and N are placed on a smooth inclined plane as shown in the figure. All surfaces of the blocks are parallel to the inclined plane. They are moving upward along the inclined plane with the same initial speed and uniform deceleration. Which of the following statement is correct ?

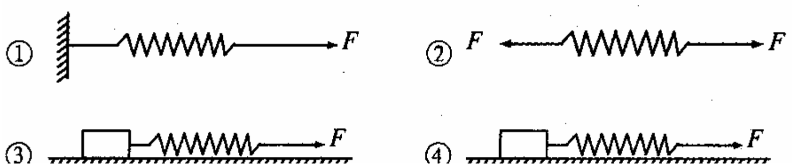
A. The friction acting on M by N is upward.
B. The friction acting on M by N is downward.
C. The friction between M and N is zero.
D. The existence of friction depends on the properties of their surfaces.



Analysis

As the inclined plane is smooth and all the surface of the blocks are parallel, they have the same acceleration $a=g\sin\alpha$ (downward). They also have the same initial speed. Thus, there is no tendency of relative motion and the friction is zero. C is correct.

Example 5



As shown in the diagram, four identical springs are placed horizontally and a force F pulls at the right end of each of the springs. In the first diagram the left end is attached to a wall. In the second diagram, the left end is acted by the same pulling force F. In the third case, the left end is attached to a small block which is placed on a smooth surface. In the forth case, the left end is attached to a small block which is placed on a rough surface. Let l_1, l_2, l_3 and l_4 are the extensions of the four springs and neglect the masses of the springs. Which of the following statements is correct ?

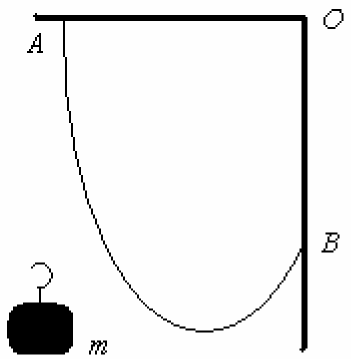
A. $l_1 < l_2$
A. $l_3 < l_4$
A. $l_1 > l_3$
A. $l_2 = l_4$

Analysis

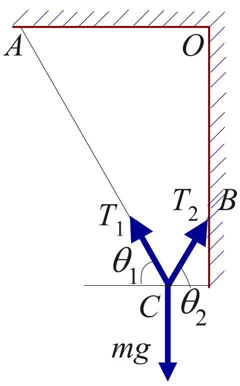
In the above four cases, the extensions depend on the force F only if the mass of the spring is neglected. As the forces F are the same, then the extensions are the same. D is correct.

Example 6

As shown in the diagram, a light and soft thin wire is attached on the ceiling A and a vertical wall at B respectively. OA and OB are the same. The length of the wire is twice of the distance OA. A mass m is now hung on the wire. What is the tension of the wire at equilibrium ? (Neglect the friction between the hook and the wire)



Analysis



The forces acting on the hook at equilibrium are shown in the diagram.

The horizontal forces are balanced : $T_1 \cos\alpha_1 - T_2 \cos\alpha_2 = 0$

The hook can be thought as a pulley. It changes the direction of the wire but not the tension. So, the tension in the wire : $T_1 = T_2 = T$. Then, $\alpha_1 = \alpha_2 = \alpha$.

The vertical forces are balanced : $T_1 \sin\alpha_1 + T_2 \sin\alpha_2 - mg = 0$.

We have, $2T\sin\alpha - mg = 0$, $T = \dfrac{mg}{2\sin\alpha}$.

By geometry, we have $OA = AC\cos\alpha + BC\cos\alpha = (AC + BC)\cos\alpha$.

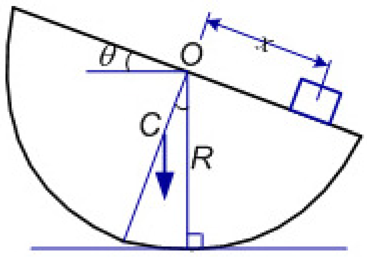
i.e. $\cos\alpha = \dfrac{OA}{AC+BC} = \dfrac{1}{2}$, $\alpha = 60^0$.

$T = \dfrac{mg}{2\sin 60^0} = \dfrac{mg}{\sqrt{3}}$.

<u>Example 7</u> (The 2nd Pan-Pearl River Delta Region Physics Olympiad Basic Contest Question 2)

(1) A uniform hemisphere with radius R is placed on a horizontal surface. Its centre of mass C is below the centre O where $OC = 3R/8$. An object is placed on the flat surface of the hemisphere. The mass of the object is 1/8 of the mass of the hemisphere. The coefficient of friction between the object and the surface of the hemisphere is μ. What is the maximum distance of the object from the centre O without slipping ?

(2) Determine the state of equilibrium (stable, neutral or unstable) when a hemisphere is placed on horizontal surface (nothing is placed on top of it). Explain briefly.



<u>Analysis</u>

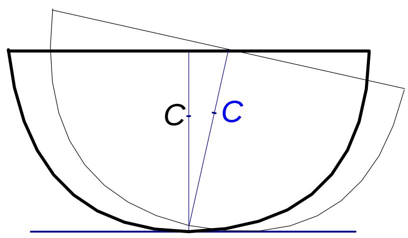
(1)

$$\sum M_O = 0,\ G\left(\frac{3R}{8}\sin\theta\right) = \frac{G}{8}(x\cos\theta),\ x = 3R\tan\theta$$

$$f = mg\sin\theta,\ N = mg\cos\theta,\ \therefore \tan\theta = \frac{f}{N}$$

$$\tan\theta_m = \frac{f_m}{N} = \frac{\mu N}{N} = \mu,\ x_m = 3R\tan_m = 3\mu R$$

(2) Stable equilibrium. It has a minimum potential when comparing with the one when it is displaced a little, because its center of gravity is lowest.



Chapter 3 Newton's Law of Motion and Law of Universal Gravitation

Isaac Newton (1642-1727) was a British scientist. He was the founder of the classical mechanics (Newtonian mechanics). He published his famous scripture "Principia" in 1687 and kicked off the new era of development in natural science. As shown in his book, a lot of research and experiments have been done based on the work of previous scientists (*e.g.* Galileo). Three laws of motion were summarized and became the foundation of classical mechanics. Newtonian mechanics is suitable for low speed macroscopic motions but not the microscopic or high speed motions. Relativity and Quantum mechanics should be used to solve the problems in microscopic world and high speed motions.

3.1 Mass

Basic concept and theory

Mass is the amount of physical matter of an object. Mass is a scalar and is one of the base unit. Its SI unit is kilogram (kg).

The inertia of an object is measured by its inertial mass. The gravitational force between two objects is measured by their gravitational masses. When a suitable unit is chosen, the inertial mass and the gravitational mass of an object are the same. They are two properties of the same object. In general, they are collectively called mass.

Points to note:

1. Mass and weight are different physical quantities. Mass is one of the intrinsic properties of an object while weight is a force acting on the object by an external object. When kg is used as the unit of force, 1 kg of mass at latitude 45° on the sea level has a weight 1 kg-force. However, its weight is not 1 kg-force at the other places.

2. Mass and energy are related. Mass can be seen as the amount of the internal energy of an object. The relation is $E = m_0 c^2$.

3. According to Relativity, the mass of an object increases with its speed (i.e. $m = \frac{m_0}{\sqrt{1-(v/c)^2}}$, where m_o is the rest mass, v is the speed of the object and c is the speed of light in vacuum.)

3.2 Newton's first law

Basic concept and theory

1. The statement of the first law

An object moves with uniform speed along a straight line or at rest unless it is acted by an unbalanced force.

2. Inertia of an object

The behaviour of an object to maintain its state of uniform motion or rest is called inertia. Newton's first law of motion is also called the law of inertia. Inertia is a basic property of all objects. Mass is a measurement of inertia of an object.

3. State of equilibrium

An object interacts with the surrounding objects. There is no object without external force acting on it. An object moving with uniform speed along a straight line or staying at rest is a result of balanced forces.

Points to note:

1. According to Newton's first law, force is not the cause of a motion. It is the cause of the change in motion (acceleration).
2. Newton's first law is the abstract conclusion from an idealization (without external force) of a real object. It is the foundation of mechanics. It discloses the inertia of objects. Inertia frame is the reference frame used throughout the mechanics.
4. Ideal experiment is also called the thought experiment. It is based on robust real experiences. It is an idealized experiment extended by scientific abstract thinking.

3.3. Newton's second law of motion

Basic concept and theory

The acceleration of an object is directly proportional to the resultant external force $\sum \vec{F}$ and is inversely proportional to its mass. The acceleration and the resultant force are in the same direction. Its mathematical expression is $\sum \vec{F} = km\vec{a}$.

The motion of an object depends on its initial state, external forces and its mass. Force is the cause of acceleration.

Points to note :

1. Using SI unit, the constant k in the expression $\sum \vec{F} = km\vec{a}$ is 1 without unit. The unit of force in SI unit is Newton (N), i.e. 1N =1kgms^{-2}. As a result, the expression becomes $\sum \vec{F} = m\vec{a}$.
2. Newton's second law is instantaneous. Force and acceleration appear, disappear and change together at every instant. When $\sum \vec{F} = 0$, acceleration is also zero at that moment.
3. Acceleration $\vec{a}$ is relative to the ground which is at rest or to an inertial reference frame

which is moving with constant speed along a straight line.

4. $\sum \vec{F} = m\vec{a}$ is a vector equation. Before applying, a specific direction should be defined as the positive direction. All the forces and accelerations should be taken as positive if they have the same direction as the positive direction and vice versa. Usually, the direction of acceleration is chosen as the positive direction. According to the principle of superposition of forces, in applying to a motion within a plane, the force can be resolved into two mutually perpendicular components. Newton's law can be applied independently to each component.

i.e. $\sum F_x = ma_x$ and $\sum F_y = ma_y$

5. Newton's second law applies to all kind of motion in macroscopic world.

6. There are two ways for determining the proportionality constant in the physical laws :

a. When all the units of physical quantities are already defined, the proportionality constant k can be determined by experiment. The kinetic friction coefficient, force constant in spring and the gravitational constant are in this catagory.

b. When the units of some physical quantities are not yet defined, the proportionality constant k can be set to some value and hence define the corresponding unit of the physical quantities. Newton's second law and Ohm's law are in this catagory.

3.4 Newton's third law

Basic concept and theory

The action and reaction forces between two objects are equal in magnitude, opposite in direction and on the same straight line.

Points to note:

1. Action and reaction forces act on different bodies. It is different from two forces acting on the same object in equilibrium. For example, a horse pulls a cart forward (action). At the same time, the cart pulls the horse backward (reaction). A common misconception is : the forward force by the horse is larger than the backward force by the cart so that the cart moves forward. It is not correct as these two forces are acting on two different bodies. The cart accelerates as the pulling by the horse is larger than the resistive force acting on it. The forward force on the cart by the horse and the backward reaction force on the horse by the cart are always the same.

2. Action and reaction forces appear and disappear together and act on different bodies. The result of the two forces may be different. For example, an egg hits a stone. The egg breaks but the stone does not. We cannot conclude that the force acting on the egg is larger than the force acting on the stone. Actually, they are the same in magnitude.

3. Action and reaction forces are the same in nature. If the action is friction, the reaction must be friction as well.

4. Newton's third law applies to the object at rest or in motion. For example, the Earth attracts

an apple. The apple attracts the Earth as well. The attractive forces are the same. The apple falls on the Earth but we cannot say that the attractive force on the apple is larger. Similarly, for a man standing in a lift, the action and reaction force between the man and the lift are always equal no matter the lift is at rest, moving with constant speed or accelerating.

5. With the help of Newton's third law, the analysis of the forces acting on one object can be transferred to the analysis of the forces acting on another one.

3.5 Gain in weight and loss in weight

Basic concept and theory

1. At equilibrium, the pressing force acting on horizontal support (or the pulling force on a string) is equal to the weight of an object. When the object accelerates in the vertical direction, the normal reaction force and its weight will not be the same.
2. When the object accelerates upward, the pressing force acting on the support will be larger than its weight. It is called gain in weight. When the object accelerates downward, the pressing force acting on the support will be smaller than its weight. It is called loss in weight. When the object accelerates downward with acceleration g, the pressing force acting on the support will become zero. It is called weightlessness. Spacecraft is performing circular motion around the Earth when it is in the orbit. It is similar to the case when the lift is accelerating downward. At this moment, the spacecraft and the astronaut are in weightless condition.
3. When using a force sensor to measure an object with mass m, the weight of it is mg. According to the ground as the reference frame, the object is acted upon by two forces : weight mg (downward) and the pulling force by the force sensor N (upward).

a. When the object is at rest or moving with constant speed along a straight line, the acceleration is zero. By Newton's second law, $N-mg = ma = 0$. Hence, $N = mg$ and apparent weight is equal to the real weight.

b. When the object is accelerating upward (the object can move upward with increasing speed or move downward with decreasing speed), then $N-mg = ma$. That is $N = mg + ma$ and $N > mg$. The apparent weight is larger than the real weight.

c. When the object is accelerating downward (the object can move upward with decreasing speed or move downward with increasing speed), then $mg-N = ma$. That is $N = mg-ma$ and $N < mg$. The apparent weight is smaller than the real weight.

d. When the downward acceleration of an object is equal to g, then $mg-N = mg$ and $N = 0$. The reading of the force sensor will become zero. The apparent weight is zero and is called weightlessness.

Points to note:

1. When an object is gaining and losing apparent weight, the weight still remains constant at that location.

2. Gaining in weight or losing in weight depend on acceleration only, not the speed.

3. In the state of weightlessness, all common effects due to the weight will be disappeared. e.g. no pendulum motion, balance does not work, no boyancy on an object in liquid, no static pressure due to liquid.

3.6 Method of separation

Basic concept and theory

This method is an effective way for analysis of forces and applying Newton's law on a problem. In using this method, draw an imaginary boundary around the object and system of objects under investigation. This helps us to see clearly the forces acting on the system under investigation.

Points to note:

In choosing the object under investigation, we can separate the whole system or part of the system from its environment. When the whole system is separated, all the internal force within the system can be neglected as they are all in pairs with equal in magnitude but opposite in direction. When a part is separated, the internal force acting on that part by the other parts in the same system should be considered as the external force acting on that separated part. Then, Newton's law can be applied.

In solving problems, we should consider the external forces acting on the separated object and its motion. Then, equations can be established according to Newton's second law. The number of unknown physical quantities should be the same as the number of equations which include the equations of motion.

Applying Newton's law to the whole system and to the separated part are complementary. They should be used simultaneously or separately to solve the problems of connected bodies effectively.

3.7 Solving problems using Newton's law of motion

Basic concept and theory

1. Two main types of problems

(1) An object with known forces : We should apply Newton's second law to find the acceleration. Together with the initial condition of motion, the motion (position, velocity with respect to time, trajectory) can be determined using the equation of motion.

(2) An object with known motion : The acceleration should be determined. Then, using Newton's second law, the forces acting on the object can be found.

2. General procedure in solving problem

a. Analyze the problem and choose a suitable object to be investigated. A suitable reference frame should be chosen. Usually the motion of the object is determined by a reference frame which is at rest relative to the ground or moving with constant speed along a straight line (Inertia reference frame).

b. Analyze all the forces acting on the object and its motion.

c. When the external forces are not on a straight and there are only two forces, the method of parallelogram can be used to determine the resultant force. If there are more than two forces, all the forces can be resolved into two mutually perpendicular components. The resultant force can then be found. Equations can be established according to Newton's second law.

d. Standardize the units in calculations. Check the answer and make necessary discussions.

3. Newton's second law consists of three physical quantities *F*, *m* and *a*. The motion of an object consists of five physical quantities v_0, v_t, *a*, *t* and *s*. The acceleration *a* connects Newton's second law and equation of motion and it's the key in solving problems in mechanics.

Forces acting on the object	$\leftrightarrow$	Newton's second law	$\leftrightarrow$	Acceleration a	$\leftrightarrow$	Equations of motion	$\leftrightarrow$	Motion of an object

$$\boxed{\begin{aligned} &F = G\frac{Mm}{r^2} \\ &G = mg \\ &F = -kx \\ &f = -\mu N \\ &F_E = Eq \\ &F_B = BIL \\ &F_B = qvB \end{aligned}} \Longrightarrow \boxed{\Sigma \boldsymbol{F} = m\boldsymbol{a}} \Longleftarrow \boxed{\begin{aligned} &a = \frac{v-u}{t} \\ &a = \frac{2(s-ut)}{t^2} \\ &a = \frac{v^2-u^2}{2s} \\ &a = r\omega^2 = \frac{v^2}{r} = 4\pi^2 rf^2 \\ &a = -\frac{kx}{m} \end{aligned}}$$

3.8 Dynamics in circular motion

Basic concept and theory

Centripetal force is named by the result of the force. It produces a centripetal acceleration and hence changes the direction of velocity of the object which is performing circular motion. The circular motion is then maintained.

The centripetal acceleration is $a = r\omega^2 = \dfrac{v^2}{r}$.

According to the Newton's second law, $|\sum \vec{F}| = ma = mr\omega^2 = m\dfrac{v^2}{r}$

Points to note:

Centripetal force is acting on an object which is performing uniform circular motion and is the resultant force of all the external forces. It is unchanged in magnitude and pointing towards the centre. Its direction changes all the time and is always perpendicular to the velocity. It does not do work on the object.

3.9 Law of universal gravitation

Basic concept and theory

Any two objects attract each other. The attractive force is proportional to the product of their masses and inversely proportional to their distance. This is the law of universal gravitation.

Mathematical expression : $F = G\dfrac{Mm}{r^2}$, where gravitational constant $G = 6.67\times10^{-11}$ N m^2 / kg^2.

Points to note:

1. The gravitational force acting on an object on the surface of the Earth can be resolved into weight and the necessary centripetal force.
2. The law of universal gravitation can apply to a wide range of astronomical phenomena. In solving these kind of problems, the weight acting on an object can be approximated to be the gravitational force acting one it, i.e. $mg = G\dfrac{Mm}{r_0^2}$, and $GM = gr_0{}^2$, where r_o is the radius and M is the mass of the Earth.

3.10 Non-inertial reference frames

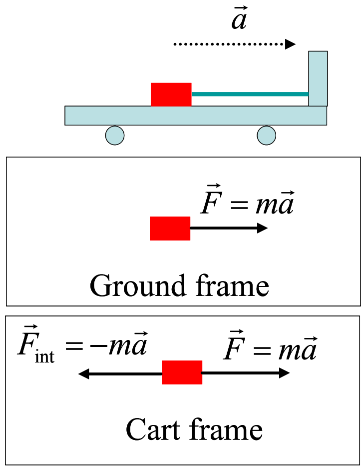
Sometimes, it is far more convenient to choose a reference frame that we know or suspect is accelerating. Recall that $\vec{a} = \vec{a}' + \vec{A}$ in Chapter-1, where $\vec{a}$ is the acceleration of the object under investigation according to an inertial reference frame S, $\vec{a}'$ is the acceleration of the object in reference frame S' that is accelerating at $\vec{A}$ relative to S. Inertial force is a 'fake' force which is present in a reference frame (say S') which itself is accelerating. Assume frame-S is not accelerating, then according to Newton's Second Law, $\vec{F} = m\vec{a} = m(\vec{a}' + \vec{A})$. So in the S'-frame, if one wants to correctly apply Newton's Law, she will get $\vec{F} - m\vec{A} = m\vec{a}'$, i. e., there seems to be an additional force

$$\vec{F}_{\text{int}} = -m\vec{A} \qquad (3.4)$$

acting upon the object.

Example

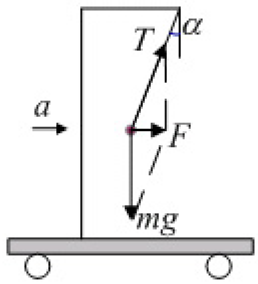
A block is attached by a spring to the wall and placed on the smooth surface of a cart which is accelerating. According to the ground frame, the force $\vec{F}$ acting on the block by the spring is keeping the block accelerating with the cart, so $\vec{F} = m\vec{a}$. In the reference frame on the cart, one sees the block at rest but there is a force on the block by the spring. This force is 'balanced' by the inertial force $\vec{F}_{\text{int}} = -m\vec{a}$



3.11 Comprehensive examples

Example 1

As shown in the figure, a right angle rigid frame is fixed on a small cart. A small ball is hung by a thread which is attached on the frame. The cart is moving to the right with acceleration $a = 2.5\text{ms}^{-2}$. How is the thread deflected and how much is the deflection ?

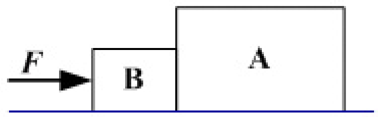


Solution

When the cart is moving to the right with uniform speed, the weight and tension of the thread acting on the ball are balanced. The thread is vertical. Now, the cart is moving to the right with uniform acceleration. According to Newton's second law, the resultant force acting on the ball is towards the right. So, the thread is deflected to the left. We have, $F = mg\tan\alpha = ma$ and $\alpha = \tan^{-1}\dfrac{a}{g} = 14°$.

Example 2

As shown in the figure, two wooden blocks A and B with different masses are placed on a smooth horizontal surface. When a horizontal force F is acting on the left side of B, they accelerate together and the force between them is N_1. When a horizontal force F is acting on the right side of A, the force between them is N_2. Determine the following statements is/are correct.



A. $N_1 + N_2 < F$ B. $N_1 + N_2 = F$ C. $N_1 + N_2 > F$ D. $N_1 : N_2 = m_A : m_B$

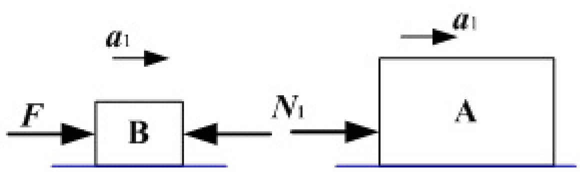
Analysis

The force between the blocks N_1 is the action/reaction pair (the pushing force on A by B and the pushing force on B by A). N_2 is similar.

Under the action of the force F, A and B accelerate to the right with acceleration a_1. According to Newton's second law, we have

$F = (m_A + m_B)a_1$, and $a_1 = \dfrac{F}{m_A + m_B}$

The forces acting on each block are shown below separately.



By Newton's second law, we have

$$F - N_1 = m_B a_1, \quad N_1 = m_A a_1 = \frac{m_A F}{m_A + m_B}$$

Similarly, when the force F acts on the left side of the block A, they accelerate to the left with acceleration a_1. Apply Newton's second law on each of them again, we have

$$N_2 = m_B a_1 = \frac{m_B F}{m_A + m_B}$$

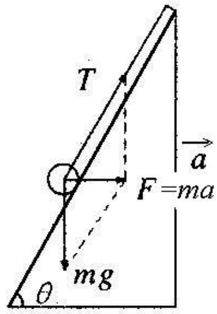
Summarize the above, we have

$$N_1 + N_2 = \frac{m_A F}{m_A + m_B} + \frac{m_B F}{m_A + m_B} = F, \quad N_1 : N_2 = \frac{m_A F}{m_A + m_B} : \frac{m_B F}{m_A + m_B} = m_A : m_B .$$

So, B and D are correct.

Example 3

As shown in the figure, a ball of mass 0.2 kg is attached to a thread and at rest on an inclined plane with an angle $\theta = 53°$. The ball is kept in contact with the surface. The thread is parallel to the surface. When the inclined plane moves to the right with acceleration 10 ms^{-2}, find the tension in the thread and the normal reaction force on the ball by the inclined plane.



Analysis:

Consider the two extreme cases : When a is small, the small ball is acted by weight, tension and normal reaction by the inclined plane. The thread is parallel to the plane. When a is large, the small ball may leave the plane. The angle between the thread and the plane is unknown.

Now, consider the critical situation in which the small ball is just left the surface, i.e. the normal reaction on the ball by the surface F_N is zero. We have

$T\cos\theta = ma, \quad T\sin\theta = mg$,

Solve the equations, we have $a_0 = g \cot\theta = 7.5$ m/s^2.

Clearly, the acceleration a = 10ms^{-2} is larger than a_o. It means that the ball is left from the surface. Then, $F_N = 0$，$T = \sqrt{(mg)^2 + (ma)^2} = 2.83\,\text{N}$.

Example 4

A car is moving on a horizontal circular path with radius R. The maximum friction between the road and the car is 1/10 of the weight. To keep the car on its track, determine the maximum speed of the car.

Analysis:

A common circular track (e.g. racing track) is higher at the outer circumference. The normal reaction can help to keep the car on its track. The track in this example is horizontal. Only the friction is responsible for the necessary centripetal force. Higher speed requires larger friction. When the speed reaches a certain value, the friction reaches maximum. The speed is the critical speed. If the speed is higher than this critical value, friction cannot be increased any more. Then, the car will not perform circular motion and will skid outward away from the track. The required equation is $\frac{mg}{10} = m\frac{v^2}{R}$, so $v = \underline{\underline{\sqrt{\frac{Rg}{10}}}}$.

Example 5

A spacecraft is orbiting on the surface of an unknown planet. In order to determine the density of the planet, which one of the following should be measured ?

A. Radius of the planet
B. The volume of the planet
C. The moving speed
D. The period of the motion

Analysis:

This is an application of basic concepts and theory. As $M = \frac{4\pi^2 R^2}{GT^2}$, we have $\rho = \frac{M}{V} = \frac{M}{\frac{3}{4}\pi R^3} = \frac{3\pi R^3}{GT^2 r^3}$, where R and r are the radius of the orbit and the planet respectively.

Since the spacecraft is close to the surface, $R = r$ and $\rho = \frac{3\pi}{GT^2}$. The answer is *D*.

Example 6

Consider an artificial satellite orbiting around the Earth with uniform circular motion. Determine whether the following statements are correct.

A. The greater the orbiting radius, the higher the speed
B. The greater the orbiting radius, the lower the speed
C. The greater the orbiting radius, the larger the necessary centripetal force
D. The greater the orbiting radius, the smaller the necessary centripetal force

Analysis

By $v = \sqrt{\frac{GM}{R}}$, B is correct. The centripetal force is equal to the gravitational force between the Earth and the satellite, i.e. $F = G\frac{Mm}{R^2}$. As m is constant, the greater the R, the smaller the centripetal force F. So, B and D are correct.

Example 7

The core occupies about 16% of the volume and 34% of the mass of the Earth. Estimate the average density of the core of the Earth up to 2 significant figures. (Given : $R_{\text{earth}} = 6.14 \times 10^3$ km , $G = 6.67\times10^{-11}$ N m^2 / kg^2)

Analysis

According to the data given in the bracket, knowledge in gravitation is needed. The weight can be approximated to be equal to gravitational force. Then, more equations and parameters can be used. Let g be the acceleration due to gravity on the surface of the Earth :

As the weight can be approximated to be equal to gravitational force, we have

$$mg = G\frac{Mm}{R^2} \quad \text{(i)},$$

where M and m are the mass of the Earth and an object respectively.

Also, $\rho = \frac{M}{V} = \frac{M}{\frac{3}{4}\pi R^3}$ (ii)

Solving (i) and (ii), we have $\rho = \frac{3g}{4\pi RG}$ (iii)

Then, $\rho = \frac{3\times 9.8}{4\times 3.14\times 6.4\times 10^6\times 6.67\times 10^{-11}}$ = 5483.3 kg/m^3.

(iv) Since $\frac{m}{M} = 0.34$, $\frac{v}{V} = 0.16$, where m and v are the mass and volume of the core respectively.

Density of the core to the density of the Earth is $\frac{\rho_{core}}{\rho} = \frac{0.34}{0.16}$ (v)

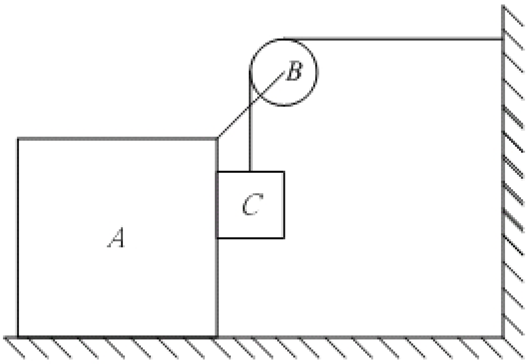
Solving (iv) and (v), the average density of the core is $\rho_{core} = 1.2\times 10^4$ kg m^{-3}

Example 8 (The 1st Pan-Pearl River Delta Physics Olympiad Contest Question 4)

In the figure, a wooden cube A of mass M is placed on a ground with coefficient of kinetic friction μ_A. B is a pulley fixed on A with negligible mass. A small wooden block C of mass m is hung by a light and inextensible thread. Block C is in contact with block A with coefficient of kinetic friction μ_C. The thread connected to block C passes through the pulley B and is fixed on the vertical wall horizontally.

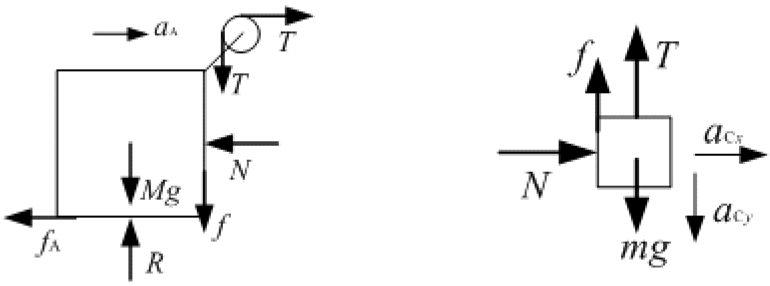
(a) Draw separately the forces diagram of the wooden block A and C. Also, and state their direction of motion.

(b) Find the acceleration of block A.



Analysis:

(a)



(b) By Newton's second law, we have

Block A: $T - N - f_A = Ma_A$, (1)

$R - Mg - T - f = 0$; (2)

Block C: $N = ma_{Cx}$, (3)

$mg - T - f = ma_{Cy}$, (4)

in which $f_A = \mu_A R,\ f = \mu_C N$ (5)

As the thread is inextensible, then $a_A = a_{Cx} = a_{Cy} = a$. (6)

Solving (1) to (6), we have $a = \dfrac{(1-\mu_A)m - \mu_A M}{(2-\mu_A+\mu_C)m + M}g$.

Example 9 (The 2nd Pan-Pearl River Delta Physics Olympiad Contest Question 3)

A wedge of mass M, inclined angle α is placed on a horizontal surface with the coefficient of kinetic friction μ. The inclined plane is smooth. A pulley is fixed on the top of the wedge. A light string passing through the pulley connects two masses m_1 and m_2 at its ends. A horizontal force F acts on the wedge M such that the mass m_1 and m_2 do not move relative to the wedge. Determine

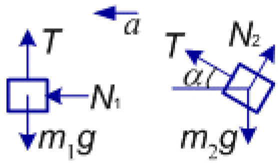
(a) The horizontal acceleration a of the system and

(b) The force F.

<u>Analysis:</u>

(a) Let the m_1, m_2 and the wedge M are relatively stay at rest initially. Now, with the force F, there is still no relative motion. So, the mass m_1 and m_2 have the same horizontal acceleration a. Also, there is no kinetic friction as there is no relative motion.

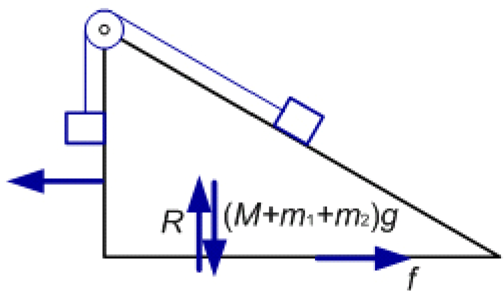
$$\sum F_y = 0, \quad T = m_1 g$$



$$\sum F_x = ma, T\cos\alpha - N_2\sin\alpha = m_2 a$$
$$\sum F_y = 0, \quad T\sin\alpha + N_2\cos\alpha = m_2 g$$

$$T\cos^2\alpha - N_2\sin\alpha\cos\alpha = m_2 a\cos\alpha$$
$$T\sin^2\alpha + N_2\sin\alpha\cos\alpha = m_2 g\sin\alpha$$

$$T = m_2 g\sin\alpha + m_2 a\cos\alpha$$
$$m_1 g = m_2 g\sin\alpha + m_2 a\cos\alpha$$
$$a = \frac{m_1 - m_2\sin\alpha}{m_2\cos\alpha} g$$



(b)

$$\sum F_x = ma, F - f = (M + m_1 + m_2)a$$
$$\sum F_y = 0, \quad R = (M + m_1 + m_2)g$$
$$F = \mu R + (M + m_1 + m_2)a = (M + m_1 + m_2)(\mu g + a)$$
$$F = (M + m_1 + m_2)g(\mu + \frac{m_1 - m_2\sin\alpha}{m_2\cos\alpha})$$

Chapter 4 Momentum and Theorem of Momentum

4.1 Momentum

Basic concept and theory

(1) Definition of momentum

The momentum of a moving object is the product of its velocity and mass. It is a physical quantity describing the state of motion of an object. i.e.

$$\vec{p} = m\vec{v} \qquad (4.1)$$

Momentum $\vec{p}$ is a vector, its direction is the same as the velocity. When two momenta are equal, they must have the same magnitude and direction.

(2) Momentum is instantaneous

Momentum is an instantaneous value at a particular moment. It changes with the instantaneous velocity.

(3) Momentum is relative

Momentum is determined relative to the reference frame. For the same object, the momenta may be different corresponding to different reference frames. The ground is chosen as the reference frame unless otherwise specified.

(4) Addition of momentum obeys the method of parallelogram.

Verification examples

Example 1

An object of mass 2.0 kg with velocity $\vec{v} = (3\vec{x}_0 - 5\vec{y}_0 + 6\vec{z}_0)\ m/s$. Calculate its momentum.

Ans : $\vec{p} = m\vec{v} = (6\vec{x}_0 - 10\vec{y}_0 + 12\vec{z}_0)$ kg$\cdot m/s$

Example 2

Determine whether the momentum will change or not in the following motions :

A. uniform speed along a straight line
B. uniform circular motion
C. an object is projected horizontally
D. an object is projected vertically

Analysis

Momentum is a vector. We should consider both of its magnitude and direction.

There are changes of momentum in B, C and D.

4.2 Impulse

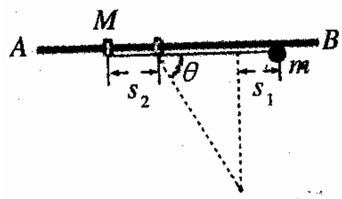
Basic concept and theory

(1) An impulse on an object due to a force is the product of the force and time of impact, i.e. $\vec{I} = \vec{F}t$. Impulse is a physical quantity describing the accumulated effect of the force with respect to time.

(2) Impulse is a vector. When the force is a constant, the direction of impulse is the same as the direction of the force. The rule of parallelogram should be used in addition of impulse.

(3) The SI unit of impulse is Newton second (Ns)

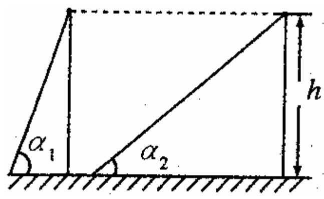
Points to note:



An impulse is the product of the force and time of impact which is independent of the motion of the object. The impulse of a changing force can be determined from the area under the curve on a *F*-*t* graph. Also, it can be calculated by the theorem of momentum.

Verification examples

Example 3



Two objects of the same mass fall from the same height of two smooth inclined planes with different angles. When they reach the bottom, they should have the same

A. impulse of the their weight
B. impulse of the normal reaction by the inclined planes
C. impulse of the resultant force when they are on the inclined plane
D. momentum
E. the horizontal components of their momentum
F. None of the above

Analysis

This example inquires the vector property of momentum. Assume the mass is *m*, the acceleration of the object when it falls along the inclined plane with the angle α_1 is a_1.

According to Newton's second law, we have $mg\sin\alpha_1 = ma_1$.

Using equation of motion , we have $t_1 = \dfrac{1}{\sin\alpha_1}\sqrt{\dfrac{2h}{g}}$, $v_1 = \sqrt{2gh}$,

where *h* is the height of the plane, t_1 and v_1 are the time and speed when it reaches the bottom.

The horizontal component of v_1 and p_1 are $v_1^{'} = v_1\cos\alpha_1$ and $p_1 = m\sqrt{2gh}\cos\alpha_1$.

Similarly, the impulse of the object on the inclined plane of the angle α_2 is

$$I_{\alpha 2} = mg\sqrt{\frac{2h}{g\sin^2\alpha_2}}.$$

The horizontal component of the momentum is $p_2 = m\sqrt{2gh}\cos\alpha_2$. As $\alpha_1 \neq \alpha_2$, then $I_{\alpha_1} \neq I_{\alpha_2}$ and $p_1 \neq p_2$. Therefore, *A* and *E* are incorrect.

As the angles are different, the normal reaction, resultant force and the speed at the bottom are different. Therefore, the impulse of normal reaction, the impulse of resultant force and momentum at the bottom are different. So, B, C, D are also incorrect.

The answer is F.

4.3 Theorem of momentum

Basic concept and theory

The impulse of the resultant force is equal to the change in momentum. It is called the theorem of momentum. An object with an initial momentum $\vec{p}_1$ is acted by an external force $\vec{F}$ for a time Δt, its momentum is changed to $\vec{p}_2$. The theorem of momentum can be expressed as

$$\vec{F}\Delta t = \vec{p}_2 - \vec{p}_1 \qquad (4.2)$$

Points to note:

1. Momentum $\vec{p} = m\vec{v}$ is a state quantity. It changes with the velocity. Impulse $\vec{F}\Delta t$ is a process quantity. The momentum of an object is changed when it is acted by impulse. Momentum and impulse are vectors. The theorem of momentum represents the impulse and the change of momentum have the same magnitude and direction. One should use the rule of parallelogram and resolving components when using the theorem of momentum.
2. For the motion on a straight line, one should specify the positive direction. The vector quantities with the same direction as the positive direction are taken as positive and vice versa. The change in momentum must be equal to the final momentum subtracted by the initial momentum. When more than one force is acting on the object, resultant is calculated from the algebraic sum of all the forces according to the sign convention of vectors.
3. When the final momentum subtends an angle with the initial momentum, the change in momentum should be determined by the rule of parallelogram or resolving components.
4. Theorem of momentum

$$\vec{F}_{\text{Resultant}}\Delta t = \vec{p}_2 - \vec{p}_1 \qquad (4.3)$$

It applies to a single object or a system of objects. Total momentum is the vector sum of all momenta of the objects in the system. Resultant force is the vector sum of all the external forces excluding the internal forces among the objects within the system. Whether a force is an external or internal force depends on the object we have chosen to investigate. For example, a shell is shot horizontally from a cannon. Excluding the friction between the cannon and the ground, the force from the gas produced by the burning explosive inside the shell is an external force both for the cannon shell and the cannon. However, when the cannon and the shell are considered as one system, the above force is an internal force. The impulse due to this internal force is zero. The total momentum of the system is kept constant.

5. The theorem of momentum is derived from Newton's second law $\vec{F}_{\text{resultant}} = m\vec{a}$ and the equation of motion $\vec{v} = \vec{v}_0 + \vec{a}t$ when the force is a constant. Newton's second law can be expressed as the resultant force is equal to the rate of change of momentum, i.e. $F_{\text{resultant}} = \dfrac{\Delta\vec{p}}{\Delta t}$. Newton's second law shows the instantaneous result while the theorem of momentum reflects the prolong effect of the resultant force. The theorem of momentum applies to both constant and varying forces. For varying forces, the force in the theorem of momentum should be the average force within the impact time. When the change in force is not in our consideration, the solution can be obtained by calculating the difference between the final and the initial momenta of the motion. This provides an alternative to solve some kind of problems that cannot be solved by Newton's second law easily. The theorem of momentum is used complementarily with the Newton's second law to solve different kind of problems in mechanics.

6. There are two main types of phenomena in applying the theorem of momentum. The first type of object is with a constant difference in momentum. When impact time is shortened, the force becomes larger and vice versa. Another kind is with a constant force. The longer the impact time, the larger change of the momentum and vice versa.

4.4 Collision

Basic concept and theory

Collision is an interaction among objects, in which, the impact time is very short and the forces are large.

Points to note:

1. During the collision, the internal forces are usually much larger than the external forces. Therefore, conservation of momentum can be used to solve problems of collision. After a head-on collision, the objects are still moving on the same straight line as the one before the collision. Otherwise, the collision is called oblique collision.

2. The total mechanical energy will usually be decreased after common collision. It can be neglected if the loss small and it is called elastic collision. The loss is maximum if the objects stick together after collision and it is called complete inelastic collision. When the objects separate but the energy loss cannot be neglected, it is called elastic collision. Normally, the total

mechanical energy would be decreased except in some cases (e.g. during the explosion, some chemical energy is changed into mechanical energy). This may help us to judge whether the answer is correct or not.

Verification examples

Example 4

As shown in the diagram, two identical balls A and B are moving within a straight line on a smooth horizontal surface. The ratio of their masses is $m_B = 2m_A$. Take the direction to the right as positive. The momentum of each of them before collision is 6kgms^{-1}. After the collision, the change in momentum of ball A is -4kgms^{-1}. Which of the following is correct ?

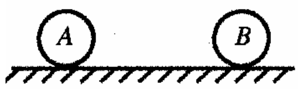
A. The ball A is on the left. The ratio of the velocities of ball A to B is 2:5 after collision.

B. The ball A is on the left. The ratio of the velocities of ball A to B is 1:10 after collision.

C. The ball A is on the right. The ratio of the velocities of ball A to B is 2:5 after collision.

D. The ball A is on the right. The ratio of the velocities of ball A to B is 1:10 after collision.

Analysis:



The change in momentum of ball A is -4kgms^{-1}. Then, the change in momentum of ball B is 4kgms^{-1}. The momenta of ball A and B after collision are $m_A v_A = 2\text{kgms}^{-1}$ and $m_B v_B = 10\text{kgms}^{-1}$.

But, we have $m_B = 2m_A$. So, $\frac{v_A}{v_B} = \frac{2}{5}$. The answer is A.

Example 5

A steel ball of mass m falls from a height and hits the ground with speed v_1. It rebounds vertically with speed v_2 after very short time of impact. What is the impulse on the steel ball by the ground during the collision ?

A. $m(v_1 - v_2)$ downward B. $m(v_1 + v_2)$ downward C. $m(v_1 - v_2)$ upward D. $m(v_1 + v_2)$ upward

Analysis:

As the impact time is very short, the impulse due to the weight can be neglected when using the theorem of momentum. In this problem, the impulse on the steel ball is due to the normal reaction by the ground only. During the calculation, the vector properties of momentum and impulse should be considered. Take upward as positive. Then, $p_1 = -mv_1$ and $p_2 = mv_2$. The impulse is $I = mv_2 - (-mv_1) = m(v_1 + v_2)$. It is positive and its direction is upward. The answer is D.

4.5 Comprehensive examples

Example 1

A newspaper in October, 1981 reported that a three-years-old boy fell from the 15th floor of a building. Luckily, he was caught and rescued by someone on the ground. Assume the impact time between the boy and his savior is 0.50s. Estimate

(1) the average impact force on his savior;

(2) the tension in the muscle of the savior; (Assume the distance from the impact point on the muscle to the elbow joint are 0.10m and 0.02m respectively.)

(3) the savior will be hurt or not. (Assume the maximum possible strength of biceps muscle is $10^8\,\text{Nm}^{-2}$ and the area of the biceps muscle when it is contracted is 50mm^2.)

Analysis

This is a bio-mechanics problem which enquires the knowledge in mechanics and biology. The key points of knowledge and application rules in each part should be clearly identified during the analysis. Transforming the real situation into models is an important way of learning physics.

(1) Assume the height of each floor is about 3m.

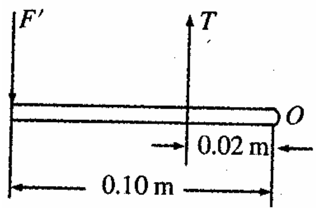
The height of the 15th floor is about $H = 15h = 45$m

The speed of the boy when he is caught is $v = \sqrt{2gH} = 30$m/s.

The mass of the boy is about 15kg. By the theorem of momentum,

$(F - G)\Delta t = m\Delta v$, then $F = G + \dfrac{m\Delta v}{\Delta t}$. And $F \approx 1050\text{N}$

(2) By Newton's third law, the average force acting on the savior F'=1050N.



By the knowledge in Biology, the forearm acts like a lever as shown in the diagram. Let the tension in the biceps muscle is T. At equilibrium, we have $F' \cdot L_1 = T \cdot L_2$. Then, T = 5020N

(3) Tensile strength $p = \dfrac{F}{A} = \dfrac{5250\text{N}}{50\times10^{-6}\text{m}^2} = 1.05\times10^{8}\text{Pa} > 10^{8}\text{Pa}$. The savior may be hurt.

Example 2

Two ice skier, each of mass 30kg, are moving towards each other with speed $v_o = 2\text{ms}^{-1}$. To avoid collision, A pushes a 15kg-box towards B as shown in the figure. What is the minimum speed of the box to avoid collision ? (Neglect friction)

Analysis:

This problem checks the knowledge in applying conservation of momentum. The choosing of the objects to be investigated and determining the critical state are crucial to the application of conservation of momentum and interactions among different bodies. We can choose A and B with the box as the system or investigate them separately. Then, the equations can be established by conservation of momentum.

The critical condition to avoid collision is :the final velocity of A after pushing the box should be the same as the final velocity of B after receiving the box.

Let the masses of A and B are M, the mass of the box is m. v_1, v2 and v are the speeds of A, B and the box relative to the ground respectively. The direction of the initial velocity of A is taken as positive.

Consider the system consisting A and the box, by conservation of momentum,

$(m+M)v_0 = mv + Mv_1$ and $(30+15)\times2=30v_1+15v$

Consider the system consisting B and the box, by conservation of momentum,

$mv - Mv_0 = (m+M)v_2$ and $15v-30\times2=45v_2$

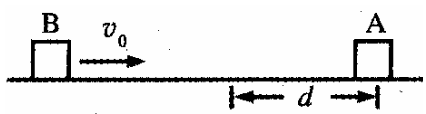
But, $v_1=v_2$. Solving the simultaneous equations, we have $v = 5.2\text{ms}^{-1}$.

If A pushes the box of a speed equal or higher than 5.2ms^{-1}, there will be no collision.

Exercise 3

For a collision between two objects, in which they move on the same straight line before and after the collision and there is no mechanical energy loss, it can be simplified using the following model. A and B are placed on a smooth surface and move on a straight only. There is no force between them when their separation is larger than a certain value *d*. When their separation is smaller than *d*, there is a constant repulsive force *F* between them.

Let *A* of mass m_1=10kg is at rest at a particular position on the straight line and *B* of mass m_2=3.0kg moves towards *A* with speed v_o. *B* is far away from A initially as shown in the diagram.



It is given that d = 0.10m， F = 0.60N，v_0 = 0.20m/s. Determine

(1) The acceleration of A and B during interaction;

(2) From the beginning of the interaction to the minimum separation of A and B, the loss in kinetic energy of the system (the group of objects);

(3) The minimum separation between A and B.

Answer:

(1) $a_1 = 0.60\text{m/s}^2$, $a_2 = 0.20\text{m/s}^2$

(2) 0.015 J

(3) 0.075 m

Exercise 4

An Eskimo dog of mass m is standing on a ski of mass M which is at rest on a horizontal icy ground. The dog jumped backward away from the ski. Then, it chased and jumped forward to the ski. The above steps were repeated and they moved along a straight line. If the dog jumped away from the ski with a velocity V, then the velocity of the dog at this moment should be V+u (u is the velocity dog relative to the ski; V+u is an algebraic sum; if the movement of the ski is take as positive, then V is positive and u is negative).The dog always chased and jumped on the ski with speed v. The friction between the ski and the icy ground can be neglected. It is given that v is 5m/s, u is 4m/s, M is 30kg and m is 10 kg.

(1) Determine the common velocity of the dog and the ski after the dog has jumped the first time on the ski.

(2) Determine the terminal speed of the ski and the maximum number for the dog jumped back to the ski.

(Two logarithmic values may be used when necessary : log2=0.301 and log3=0.477)

Answer:

(1) V_1' = 2m/s

(2) 3 times

(3) 5.625 m/s

Chapter 5 Work, Mechanical Energy and Theorem of Kinetic Energy

5.1 Work and power

Basic concept and theory

When an object is acted by a force and it moves along the direction of the force, work is done on the object. Force and displacement along the direction of the force are two crucial factors of work done.

(1) General expression of work

$$W \equiv \vec{F} \bullet \vec{S} = FS\cos\alpha \qquad (5.1),$$

where the force $\vec{F}$ along the displacement $\vec{S}$ should be a constant and α is the angle between them.

When $\alpha = 90°$, the force $\vec{F}$ does not do any work. When $0° \le \alpha \le 90°$, the force does positive work. When $90° \le \alpha \le 180°$, the force does negative work (or the opponent force does work).

(2) Work is a scalar

Positive work means the work done by the force that causes the movement and the object gains energy. Negative work means the work done by a resistive force of the movement and the object loses energy.

(3) Work done by the resultant force

There are two ways to calculate the work done by a resultant force when more than one force are acting on the object :

(a) The rule of parallelogram or resolving components is used to find out the resultant force $F_{\text{resultant}}$. Then, calculate the work done by $W = Fs\cos\alpha$, where α is the angle between the resultant force and the displacement.

(b) Calculate the work done by each force: $W_1 = F_1 s\cos\alpha_1$, $W_2 = F_2 s\cos\alpha_2$, $W_3 = F_3 s\cos\alpha_3$,… Then, the work done by the resultant force is the algebraic sum : $W = W_1 + W_2 + W_3 + \cdots$

(4) Power

Power is the ratio of the work over time for doing this amount of work. It is a physical quantity expressing the rate of doing work. Expression :

(a) $P = \dfrac{W}{t}$, it is the average power of the object within the time t.

(b) $P = \vec{F} \cdot \vec{v} = Fv\cos\alpha$: It is the average power when v is the average velocity. It is the instantaneous power if v is the instantaneous velocity. (α is the angle between F and v)

The power stated on the label of a generator is the maximum output power when it operates normally. The actual output power of the generator may not be the same as the rated value. It may vary between zero and the rated value. The power of the generator is the same of the power of the towing force. From $P = Fv$, the towing force is inversely proportional to the velocity when the output power is kept constant.

<u>Points to note:</u>

1. The difference between impulse and work : They are both quantities of process. Impulse expresses the accumulated effect on an object by a force after a period of time. Work expresses the accumulated effect on an object by a force after it has displaced. The result of the impulse is the change in momentum while the result of the work is the change in energy. Impulse is a vector while work is a scalar.
2. If the work is difficult to be determined by force and displacement, the work can be found by the change in energy as they are equal in magnitude.
3. Work done by a varying force : There are two types of forces. One is forces related to potential energy, e.g. gravity, spring force, boyancy and electric force. The work by them is independent of the path and it only depends on the initial and final positions. The potential energy of the object decreases when the force does positive work. The potential energy of the object increases when the object opposes the force. Hence, the work can be determined from the change in potential energy.

Kinetic friction, air resistance and elastic force of plastic deformation are examples of another kind of forces. When the motion is along a curve or a loop, the work of this kind of forces is the product of the force and the path (not the displacement). The work can also be determined by conservation of kinetic energy or total mechanical energy in case of varying force.

4. The difference in constant power and uniform acceleration at the start of a vehicle

a. The process of the starting motion with a constant power output of a vehicle : acceleration decreases to zero. During this process, the quantities vary as follows:

$$\boxed{\begin{array}{l}\text{velocity } v\uparrow \\ F=\dfrac{P}{v}\downarrow\end{array}} \Rightarrow \boxed{a=\dfrac{F\downarrow -f}{m}\downarrow} \Rightarrow \boxed{\begin{array}{l}\text{When } a=0, \text{then } F=f,\\ v=v_{\text{m}}\ \text{(max)}\end{array}} \Rightarrow \boxed{\begin{array}{l}\text{constant speed}\\ v_{\text{m}}\end{array}}$$

|← – – – varying acceleration – – – →| ← – – – – – – – uniform motion – – – – – – – →

The vehicle reaches its maximum velocity when $a = 0$. At this moment, $F = f$，$P = F\cdot v_{\text{m}} = fv_{\text{m}}$.

b. The process of the starting motion with a uniform acceleration :

uniform acceleration (a = constant) ➔ acceleration decreases ➔ uniform motion (a = 0)

In this process, P, F, v and a vary as follows:

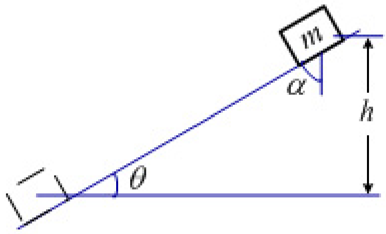
$$\boxed{\begin{array}{l}a=\dfrac{F-f}{m}\\ F, a\ \text{constant}\end{array}} \Rightarrow \boxed{\begin{array}{l}P=Fv,\\ P\uparrow, v\uparrow\end{array}} \Rightarrow \boxed{\begin{array}{l}\text{If } P=P_{\text{rated}}v,\\ a=\dfrac{F-f}{m}\neq 0\\ v\ \text{keeps increasing}\end{array}} \Rightarrow \boxed{\begin{array}{l}F=\dfrac{P_{\text{rated}}}{v\uparrow}\downarrow\\ a=\dfrac{F-f}{m}\downarrow\end{array}} \Rightarrow \boxed{\begin{array}{l}\text{If } a=0, v \text{ is max}\\ v_{\text{m}}\ \text{is a constant}\end{array}}$$

|← – – uniform acceleration – →|← – – – varying acceleration – – →| ← uniform motion – →

Verification examples

Example 1

As shown in the figure, an object of mass *m* slides down from rest on a smooth inclined plane of angle θ. When it has fallen a vertical distance *h*, what are the instantaneous power at that moment and the average power during the process of the work done by the gravity ?



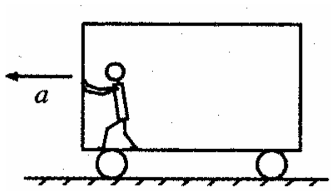
Solution:

Instantaneous power $P = Fv\cos\alpha = mgv\sin\theta = mg\sqrt{2gh}\sin\theta$

Average power $P = F\bar{v}\cos\alpha = mg\left(\frac{1}{2}\sqrt{2gh}\right)\sin\theta = mg\frac{\sqrt{gh}}{2}\sin\theta$

Example 2

As shown in the figure, a man pushes forward on the compartment which is accelerating uniformly to the left. If the man stays at rest relatively to the compartment, which of the following is correct ?



A. The man does positive work on the compartment.
B. The man does negative work on the compartment.
C. The man does zero work on the compartment.
D. It cannot be determined.

Solution:

The man and car accelerate uniformly together. Consider the man, there are two horizontal forces acting on the man : the normal reaction by the vertical wall towards the right F_1 and the friction by the floor towards the left F_2. As $F_1 < F_2$, the car does positive work on the man. Hence, the man does negative work on the compartment. The answer is B.

Example 3

The engine of a car of mass 2000kg has a normal rated power 80kW. It experiences a constant resistive force which is 1/5 of its weight when it is travelling on a straight road. Now, this car accelerates from rest at 2m/s^2, then calculate

(1) the duration of uniform acceleration;
(2) the maximum velocity reaches during the uniform acceleration;
(3) the maximum velocity reaches finally;
(4) the actual power of the car at the end of the 3^{rd} second.

Solution

Understanding clearly the motion of a motorized vehicle in different stages, clarifying the difference between actual power and normal rated power are crucial in solving this problem.

From $F - kmg = ma$,

we have $\frac{P}{v} - kmg = ma$, $\frac{P_{\text{rated}}}{v_1} - kmg = ma$, $v_1 = \underline{\underline{10\text{ m/s}}}$, $t = \frac{v_1}{a} = \underline{\underline{5\text{ s}}}$

From $\frac{P}{v} - kmg = ma, a = 0$,

we have $\frac{P_{\text{rated}}}{v_2} = kmg$, $v_1 = \underline{\underline{20\text{ m/s}}}$

When t = 3s , $v = at$, $P = Fv$ = $\underline{\underline{48\text{ kW}}}$

5.2 Mechanical energy and theorem of kinetic energy

<u>Basic concept and theory</u>

(1) Kinetic energy : The energy due to movement of the object is called kinetic energy.

$$\text{Expression} : E_k = \frac{1}{2}mv^2 \quad (5.2)$$

Unit : SI unit of kinetic energy is Joule (J)

Kinetic energy is a physical quantity describing the state of the object's motion. The difference between the kinetic energy and momentum should be noted. Kinetic energy is a scalar while momentum is a vector. The change in kinetic energy is a result of work while the change in momentum is a result of impulse. Kinetic energy and momentum are related as :

$$E_k = \frac{p^2}{2m}, \; p = \sqrt{2mE_k}$$

(2) Theorem of kinetic energy : The change in kinetic energy of an object is equal to the total work done on the object by the external forces. This is called the theorem of kinetic energy.

Equation : $W_{\text{total}} = E_{\text{k2}} - E_{\text{k1}}$.

The theorem of kinetic energy can be applied to single object. The total work done by the external forces is the same as the work done by the resultant force and the algebraic sum of all the work done by the external forces. The procedures of using theorem of kinetic energy :

(a) Choose an object to investigate and determine its motion

(b) Analyze the external forces acting on the object and work done by each force

(c) Determine the initial and final kinetic energy of the object

(d) State the expression of the theorem of kinetic energy and other related equations for solving the problem

When the velocity changes uniformly on a straight line and the problem does not enquire the acceleration or time, using the theorem of kinetic energy is simpler than Newton's law of motion in general. The theorem of kinetic energy can also be used to solve the kind of problems (e.g. the process of the action of a varying force, motion on a curve) which cannot be solved by Newton's law easily.

(3) Potential energy

(a) The energy which is defined by the interaction and relative position is called potential energy, e.g. gravitational potential energy, elastic potential energy and electric potential energy.

(b) Gravitational potential energy : An object being raised possesses gravitational potential energy. This energy is possessed by both the object and the Earth. It is defined by the relative position of the object to the Earth.

Equation : $E_p = mgh$, gravitational potential energy is a scalar

Unit: The SI unit is Joule (J)

The gravitational potential energy is relative. In the equation $E_p = mgh$, where h is the height of the centre of gravity of the object from a reference height (the height of zero gravitational potential energy). When the object is above this reference height, it has positive gravitational potential energy. When the object is below this reference height, it has negative gravitational potential energy. The change in gravitational potential energy is independent of the zero potential energy reference height.

The change in gravitational potential energy is due to the work done by the gravity force. The positive work done by the gravity force is equal to the loss in gravitational potential energy. The negative work done by the gravity force is equal to the gain in gravitational potential energy. Hence, the work done by the gravity force is equal to the negative of the change in gravitational potential energy. i.e.,

$$W_G = -(E_{p2} - E_{p1}) = E_{p1} - E_{p2} \quad (5.3)$$

The work due to gravity force depends on its initial height h_1 and final height h_2 of the object. It is independent of the path of the object.

(d) ©Elastic potential energy : A distorted elastically object will do work on its environment when it is restoring its original shape. Hence, it possesses energy and it’s called elastic potential energy.

$$\Delta E = \frac{1}{2}k(x_2^2 - x_1^2) \qquad (5.4),$$

where k is force constant, x_2 and x_1 are displacement.

(4) Mechanical energy : The kinetic energy, gravitational potential energy and elastic potential energy are collectively called mechanical energy.

Points to note:

(1) Determine the work done by a varying force using the theorem of kinetic energy :

In some cases, the work done by F could not be determined by the expression $W = Fs \cos\alpha$ due to the varying magnitude or direction of the force F. The work done by the force F can be now determined by the result of the work, i.e. the change in kinetic energy.

(2) If there are different types of motion in the whole process (e.g. accelerate, uniform velocity, decelerate), we can consider each type of motion separately or the whole process when applying

the kinetic energy theorem. The problem may be simplified if the equation is set up for the whole process. When the work done each force is being substituted into the equation, the work should carries its value and sign. The work due to each force should be clarified at the beginning.

$$W_1 + W_2 + \cdots + W_n = \frac{1}{2}mv_t^{\,2} - \frac{1}{2}mv_0^{\,2} \qquad (5.5)$$

Verification examples

Example 4

A block of mass 200g at a height 10m is projected horizontally with a speed 10m/s.

(1) Calculate the speed of block when it hits the ground. (Neglect air resistance)

(2) Due to air resistance, the speed of the block when it hits the ground is now 15m/s. What is the work done against the resistive force ?

Solution:

In solving this problem, we should analyze the energy flexibly, grasp the condition of using conservation of mechanical energy and understand accurately a basic concept of work, i.e. work is a measure of energy transfer.

(1) $mgh = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^{\,2}$, $v = \underline{\underline{17.3 \text{ m/s}}}$

(2) Method 1

$W_G - W_f = \frac{1}{2}mv_2^{\,2} - \frac{1}{2}mv_0^{\,2}$, $mgh - W_f = \frac{1}{2}mv_2^{\,2} - \frac{1}{2}mv_0^{\,2}$, $W_f = \underline{\underline{7.1 \text{ J}}}$

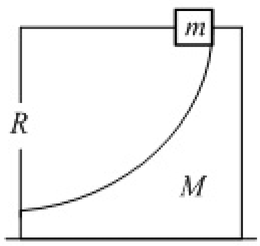
Method 2

$W_f = mgh - \frac{1}{2}mv_2^{\,2} + \frac{1}{2}mv_0^{\,2} = \underline{\underline{7.1 \text{ J}}}$

5.3 Comprehensive examples

Example 1

As shown in the diagram, a small block of mass *m* is released from rest at the top of a smooth track which is on a large block. The large block of mass *M* is at rest on a smooth surface. The track forms a quarter of a circle with radius *R*. The small block slides from the top to the bottom of the track where the tangent is horizontal. Determine the work done by the normal reaction on the small block in the whole process.



Solution:

The large block M recedes during the interaction with the small block m. Hence, the normal reaction F_N is not perpendicular to the movement of the small block m. Work is done by F_N. F_N is varied in magnitude and direction. It is difficult to determine the work done by F_N directly. Instead, the whole process is governed by two rules. One is the conservation of mechanical energy of the system consisting of m, M and the Earth. Another one is the conservation of momentum along the horizontal direction of the system of m and M. In addition of applying the theorem of kinetic energy on m. The work due this varying force can be determined.

Let v be the speed of m when it reaches the bottom, V be the receding speed of M :

$$mgR = \frac{1}{2}mv^2 + \frac{1}{2}MV^2$$

$$mv = MV$$

$$W_N + mgR = \frac{1}{2}mv^2$$

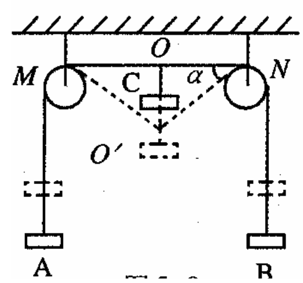
After solving, we have $W_N = -\dfrac{m}{m+M}mgR$

Example 2

Two identical masses A and B are connected by a thread which passes through the pulleys M and N as shown in the diagram. The masses are at rest initially and the distance between the pulleys is 0.6m. Now, a weight C of the same mass is hung gently in the mid-point of MN.

(1) Calculate the distance moved when C reaches its maximum speed.

(2) What is the maximum distance moved by C ?



Solution:

Conservation of mechanical energy is used to investigate a many bodies system. One should investigate the forces acting on each object, the work done by each force and the mechanical energy transfer among the objects. The gain in mechanical energy of an object is equal to the loss in mechanical energy of another object, i.e. $E_{gain}=E_{loss}$. This may help us to establish the necessary equations. The last expression in part (2) demonstrates the loss in gravitational potential energy of C is equal to the gain in gravitational potential energy of A and B.

(1) The masses A, B and C accelerate at the beginning and then decelerate. The resultant force acting on each of them is zero when the speed of C is maximum (C is at position O' at this moment). If their masses are the same, then $\alpha = 30°$ and the distance moved by C is

$$h = \frac{0.6}{2} \times \tan 30^0 = \underline{0.17 \text{ m}}$$

(2) When C reaches the lowest position, A and B reach their highest position. Their speeds are zero.

Let H be the maximum distance moved by C:

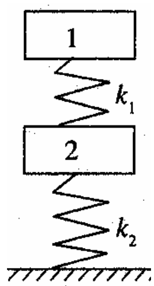
The distance moved by A and B is $\sqrt{h^2+0.3^2}-0.3$.

Only the weight does work on the system of A, B and C and the mechanical energy is conserved.

$$mgH = 2mg(\sqrt{h^2+0.3^2}-0.3)\,,\ H = \underline{\underline{0.4\text{m}}}$$

Example 3

As shown in the diagram, a spring of spring constant k_1 connects the blocks 1 and 2 of mass m_1 and m_2 respectively. A spring of force constant k_2 connects at the bottom of the block 2 and presses on the ground.



The system is at equilibrium. Now, block 1 is lifted slowly until the spring at the bottom has just left the ground. Determine the increase in gravitational potential energy of blocks 1 and 2.

Analysis:

This problem enquiries the analysis of contextual physics. One should determine the change of states of the connected objects according to their initial and final states.

Initially, the springs are at equilibrium.

Consider m_2 : we have $k_2x_2 = (m_1+m_2)g,\quad x_2 = \dfrac{(m_1+m_2)g}{k_2}$

Consider m_1 : we have $k_1x_1 = m_1g,\quad x_1 = \dfrac{m_1g}{k_1}$

Finally, the springs are at new equilibrium.

The extension of the spring k_1 is $x_1' = \dfrac{m_1g}{k_1}$.

From the above,

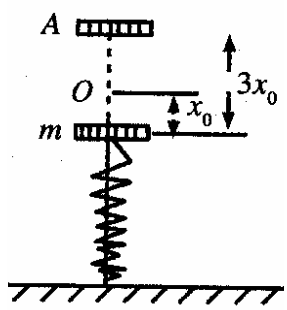
The increase in gravitational potential energy of block 2 is

$$m_2gx_2 = m_2g\frac{(m_1+m_2)g}{k_2} = \underline{\underline{m_2(m_1+m_2)g^2\frac{1}{k_2}}}$$

The increase in gravitational potential energy of block 1 is

$$m_1g(x_1+x_1'+x_2) = \underline{\underline{m_1(m_1+m_2)g^2\left(\frac{1}{k_1}+\frac{1}{k_2}\right)}}$$

Example 4



A steel plate of mass m is connected to the upper end of a spring which is fixed on the ground. At equilibrium, the compression is x_o as shown in the diagram. An object is fallen from rest at a height $3x_o$ above the original one. It hits on the lower plate and moves downward together but are not attached. They moves upward after reaching the lowest position. It is given that they can just reach the point *O* if the mass of the object is *m*. When the mass of the object becomes $2m$. they possess some upward velocity at the point *O*. Find the distance between the maximum point they can reach and the point *O*.

Solution:

This problem enquires the integration of conservation of mechanical energy and conservation of momentum, analysis and deduction. By the analysing the physical process, the elastic potential energy can be deduced artfully. It conforms with the requirement of mathematics in the competition. The physical process should be analysed seriously. Then, the equations can be constructed correctly.

From the equations of free falling, the velocity of the object before hitting the plate is $v_1 = \sqrt{2gh} = \sqrt{6gx_0}$. During the collision, the momentum is conserved. Thus, the common velocity after collision is $v_2 = \frac{1}{2}v_1 = \sqrt{\frac{3gx_0}{2}}$.

The mechanical energy is conserved when they compress the spring and moves upward to the point *O*. Let the initial elastic potential of the spring is E_p and the point *O* is the zero gravitational potential. Thus, $E_p - 2mgx_0 + \frac{1}{2}(2m)v_2^{\ 2} = 0$. We have $E_p = \frac{1}{2}mgx_0$.

The object of mass $2m$ with the same velocity $v_1 = \sqrt{6gx_0}$ before hitting the lower plate. Momentum is conserved during collision. Let v_2' be the common velocity after collision. Then, we have $2mv_1 = 3mv_2'$ and $v_2' = \frac{2}{3}v_1 = \sqrt{\frac{8gx_0}{3}}$.

The mechanical energy is conserved when they compress the spring and moves upward to the point *O*. Let the velocity when they pass through the point *O* is v_3 and the point *O* is zero gravitational potential. Thus, we have $E_p - 3mgx_0 + \frac{1}{2}(3m)v_2^{\ 2} = \frac{1}{2}(3m)v_3^{\ 2}$ and $v_3 = \sqrt{gx_0}$.

The object of mass $2m$ is projected vectically with speed v_3 from the point *O*. The maximum height it can reach is $h = \frac{v_3^{\ 2}}{2g} = \underline{\underline{\frac{x_0}{2}}}$.

Chapter 6 Laws of Conservation in Mechanics

In the ever changing nature, we can still find that some of the physical quantities (mechanical energy, momentum, centre of mass, charge, etc.) are conserved under conditions. Some of the laws of conservation have a wide range of applicability like conservation of momentum. Thus, searching for conservation of physical quantities has its unique importance in Physics. Law of conservation is also an important method for solving problems in Physics. Some of the well known laws of conservation in mechanics are stated below with examples.

6.1 Conservation of mechanical energy

Basic concept and theory

(1) Law of energy conservation

a. Energy cannot be destroyed or created.

b. Energy can be transferred from a form to another form with the total energy unchanged.

(2) Law of mechanical energy conservation

When work is done by gravitation force or/and elastic force by a spring only, the kinetic energy interchanges with its potential energy. The total mechanical energy is kept constant.

General expression : $E_1=E_2$ or $E_{k1} + E_{p1} = E_{k2}+ E_{p2}$;

Anther form : $\Delta E_k= -\Delta E_p$ or $\Delta E_k+\Delta E_p=0$.

(3) Two types of conservation of mechanical energy

When work is done by gravitation force or/and elastic force by a spring only on an object, the total mechanical energy is conserved. There is no work done by the other forces or the algebraic sum of the work done by the other forces is zero.

The mechanical energy of a sytem is conserved under the following conditions:

There are only interchanges between kinetic and potential energy among the objects in a system ;

There is no interchange of mechanical energy with the environment;

The mechanical energy does not change to other form of energy (e.g. internal energy).

(4) Basic steps for applying law of conservation of mechanical energy :

Step 1 : Choose a target to be analyzed (an object or system of objects) according to the problem

Step 2 : Analyze the forces acting on the target and the work done by the forces and see whether the mechanical energy is conserved.

Step 3 : Choose a suitable reference plane and determine the initial and final mechanical energy including the kinetic and gravitational potential energy.

Points to note:

1. Work is a measure of transfer of enegy. It is equal to the amount of energy transferred. On another hand, the amount of energy transferred is the work. This is another view of the conservation of energy.

By the expression of work, the work done by a force other than the weight and elastic force is equal to the change in mechanical energy.

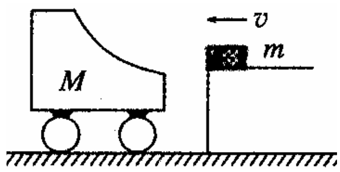
$$\sum W = E_2 - E_1 = \Delta E \text{ or } W_{\text{motive force}} + W_{\text{resistive force}} = E_2 - E_1,$$

i.e. $Fs - fs = (\frac{1}{2}mv_2{}^2 + mgh_2) - (\frac{1}{2}mv_1{}^2 + mgh_1)$

The change in mechanical energy in a system depends on the initial and final states during the process of doing work and it is independent of the process. The difficulties of determining the acceleration in Newton's second law can be avoided when the work-energy relations and conservation of mechanical energy are used for solving problem.

2. The total work done against friction is equal to the loss in mechanical energy. This amount of energy is transferred into internal energy of thte system. Friction causes an increase in temperature. Thus, the product of kinetic friction acting on the system and the relative displacement between objects inside the system is equal to the increase in internal energy (i.e. $Q = f_{\text{sliding}} \times s_{\text{relative}}$). The change in internal energy due to static friction is zero as the relative displacement is zero even when it has work done.

Example 1



A trolley of mass M is placed on a smooth horizontal plane. A small metal block of mass m is sliding on a horizontal trough with velocity v as shown in the diagram. Determine

(1) The maximum height H that the metal block can reach. (Assume the metal block will not slip away from the left of the trolley.)

(2) The maximum speed of the trolley v_2.

Solution

(1) When m reaches its maximum height and moves with M of a common velocity, it is an inelastic collision. i.e. By conservation of momentum, we have $mv = (m+M)v'$.

When m is sliding up on the smooth trough, the mechanical energy is conserved : $\frac{1}{2}mv^2 = mgH + \frac{1}{2}(m+M)v'^2$.

Solving, we have $H = \dfrac{Mv^2}{2g(m+M)}$.

(2)When m is receding, M is still accelerating. When m is left from the right of the trolley, the velocity of M is maximum. By conservation of momentum and energy, we have

$$mv = mv_1 + Mv_2$$

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_1{}^2 + \frac{1}{2}Mv_2{}^2$$

From the above, we have $v_1 = \frac{m-M}{m+M}v, \quad v_2 = \frac{2m}{m+M}v$.

6.2 Conservation of momentum

Basic concept and theory

The total momentum of a system of interacting objects is conserved when there is no external force and the resultant external forces is zero. It is expressed as :

$m_1v_1 + m_2v_2 = m_1v'_1 + m_2v'_2$ or $p_1 + p_2 = p'_1 + p'_2$ ， $p = p'$ 。

Points to note :

1. Law of conservation of momentum is one of the important and common laws in nature. It can be checked experimentally or derived from the theory of momentum and Newton's third law. It can be known as a particular case of the theory of momentum.
2. The conditions of using conservation of momentum:

a. There is no external force and the resultant external forces is zero.

b. When the resultant of the external forces is not zero and which is much smaller than the internal force within the system (e.g. the friction during the collision, the weight during explosion), the external forces can be neglected.

c. When the resultant of the external forces is not zero but which has zero component on one particular direction, the component of the total momentum along that particular direction is thus conserved.

3. The applicability of law of conservation of momentum

a. The law of conservation of momentum has a wider range of applicability than the Newton's law of motion. It can apply to a system of which there is no external force or the total resultant force is zero no matter the interactions are gravitational force, elastic force, frictional force or electromagnetic force.

b. It can apply to a system whether the objects are in contact or not, are sticked together or exploded into pieces after interaction. It can also be applied to macroscopic systems like galaxies, microscopic systems like atoms and fundamental particles, problems of the objects with low speed and the objects with speed near the speed of light.

c. In calculating of momenta, the velocity of each object must be referred to the same inertia reference frame. The ground is taken as the reference frame in common.

Example 2

A boat of mass *M* is at rest on a static water surface. A man in the stern walks from rest to the bow with velocity v_1 relative to the boat. If the mass of the man is *m*, determine the velocity of the boat relative to the water.

Solution

The static water is taken as the reference frame. Assume the velocity of the man and the boat relative to the water are v_2 and v_3 respectively. The direction of movement of the man is taken as positive. Thus, v_2 is positive. v_3 is unknown and assumed to be positive. We have

$mv_2 + Mv_3 = 0$.

$\because v_2 = v_1 + v_3$, $\therefore v_3 = -\dfrac{mv_1}{m+M}$, the direction of v_3 is opposite to the movement of the man. This is a standard man-boat model.

If v_3 is taken as negative, we have $mv_2 - Mv_3 = 0$, $\because v_2 = v_1 - v_3$, $\therefore v_3 = \dfrac{mv_1}{m+M}$. Thus, the solution is an absolute value as v_3 was taken as negative. The two answers are consistent. The speed is $\dfrac{mv_1}{m+M}$.

If the equation of conservation of momentum is stated as $mv_1 + Mv_3 = 0$, a mistake is done as the velocities did not refer to the same inertia frame.

Example 3

A small ball *A* of mass 0.1kg (m_1) moving to the right with speed v_1=30m/s on a smooth horizontal surface. The ball *A* then collides with a ball *B* of mass 0.5 kg (m_2) which is moving to the left with speed v_2=10m/s. Ball *A* is rebounded with the same speed as before. Find the magnitude and direction of the velocity of ball *B* after collision.

Solution

This is a direct application of the law of conservation of momentum. The initial moving directions of the two objects are different. We can define the positive direction and use positive and negative sign for representing the direction of movement. Let the direction of v_1 (to the right) as positive. The velocities of the objects can be expressed as :

$v_1 = 30$m/s，$v'_2 = -10$m/s，$v'_1 = -30$m/s。

By conservation of momentum,

$$m_1v_1 + m_2v_2 = m_1v'_1 + m_2v'_2$$

$$v'_2 = \frac{m_1v_1 + m_2v_2 - m_1v'_1}{m_2} = 2 \text{ m/s}$$

Then, the velocity is 2m/s to the right.

6.3 Comparison of two laws of conservation

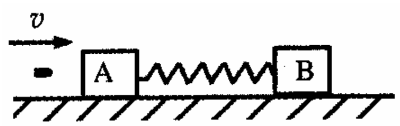
Basic concept and theory

The target under investigation of laws of conservation of momentum and conservation of mechanical energy is system consisting interacting bodies. They both investigate a physical process but the conditions for conservation are different. The momentum is conserved if the the resultant of the external forces is zero while the mechanical energy is conserved if the forces other than the weight (either internal or external forces) have not done any work.

Thus, we should stress on the work done by forces (except the gravitational force) in solving problems by the law of conservation of mechanical energy. We should stress on external forces acting on the system and see whether the resultant is zero when using the law of conservation of momentum.

Points to note:

When the momentum of a system is conserved, its mechanical energy may not be conserved and vice versa. This is the result of the different conditions of the two conservation laws.



As shown in the diagram, wooden block *A* and *B* are connected by a spring and placed on a smooth horizontal surface. A bullet is shot into the block A horizontally and stays in it. Consider the bullet, the wooden blocks and the spring as a system. To determine whether the momentum and mechanical energy are conserved or not, we should consider the conditions for the laws of conservation. Work is done against the kinetic friction during the bullet is immerging into *A*. A potion of mechanical energy of the bullet is transferred into internal energ. Thus the mechanical energy of the system decreases and is not conserved. During the process, there is no external force acting on the system in the horizontal direction. Hence, the momentum is conserved. Also, momenta are conserved in different types of explosion, collision or recoil when $F_{\text{internal}} >> F_{\text{external}}$. In many situations, work is done by the internal forces and hence, the mechanical energy is not conserved.

Also, the law of momentum conservation is a vector expression. Direction should be considered in application. It can be applied to a particular direction only. The law of conservation of mechanical energy is a scalar expression. It is the algebraic sum of work done or energy and cannot be resolved into components.

6.4 The conservation law of centre of mass

Basic concept and theory

(1) Motion of the centre of mass

A system is consisted of several point masses m_1, m_2… and they are at the positions (x_1, y_1), (x_2, y_2), … respectively. The centre of mass C (x_C , y_C) of the system is

$$x_C = \frac{\sum_i m_i x_i}{\sum_i m_i}, \quad y_C = \frac{\sum_i m_i y_i}{\sum_i m_i}.$$

e.g. The positions of the point masses m_1 and m_2 are (x_1, y_1), (x_2, y_2) respectively. The centre of mass is (x_C , y_C), where

$$x_C = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}, \quad y_C = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}.$$

(2) The conservation law of centre of mass

If there is no external force acting on the system in a particular direction, the position of the centre of mass in that direction is unchanged.

Example 6

As shown in the diagram, a small ring of mass M is hung on a smooth horizontal rod AB. A small ball of mass m is connected to the small ring by a string of length L with negligible mass. Now, the string is stretched straight and is parallel to AB. The ball is released from rest. What is the displacement of the ring s_2 when the AB and the string subtend an angle θ ?

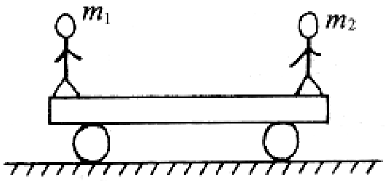
Solution

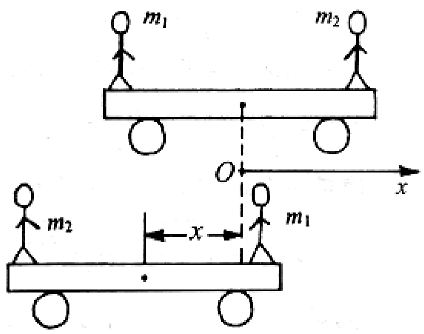
Draw a displacement-velocity graph. The problem can be solved in accordance with (a) the total momentum of the system is zero and (b) Momentum is only conserved in the horizontal direction. Thus, we have $0 = mv_1 - Mv_2$. The horizontal displacement of the two objects are related by $s_1 + s_2 + L\cos\theta = L$.

From the above two expressions and the displacement-velocity graph, we have $\frac{s_1}{s_2} = \frac{M}{m}$. After solving, we have $s_2 = \frac{mL(1-\cos\theta)}{M+m}$.

Example 4

A flat cart of mass M and length $2l$ is rest on a smooth horizontal surface as shown in the diagram. There are two men of masses m_1 and m_2 at its ends where $m_1 > m_2$. Now, they move towards each other and swap their positions. What is the distance moved by the cart ?





Solution

As shown in the diagram, a system of two men and a flat cart is rest on a smooth surface. As there is no horizontal external force, the centre of mass is unchanged even they have swapped. The original mid-point of the cart as the origin of our coordinate system. Then, we have $x_c = x'_c$ where x_c and x_c' are the centres of mass of the system before and after they have swapped.

As the external resultant force is zero, according to the conservation law of centre mass $F = ma_c$, the system is at rest (a_c=0). The centre of mass is unchanged. We have

$$\frac{m_1(-l)+m_2 l}{M+m_1+m_2} = \frac{M(-x)+m_2(-l-x)+m_1(l-x)}{M+m_1+m_2}$$

Solving, we have $x = 2l \times \dfrac{m_1 - m_2}{M+m_1+m_2}$.

6.5 Examples

Example 1

Two balls of masses m_1 = 4kg and m_2 = 1kg are moving each with speeds u_1 = 1ms^{-1} and u_2 = 2ms^{-1} respectively. They are kept moving on the same straight line after collision. Find the velocities of them if

(1) the collision is perfectly elastic collision;

(2) 25% of mechanical energy is lost during the collision.

Solution

(1) By the conservation of momentum $m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$,

By the conservation of mechanical energy $\frac{1}{2}m_1 u_1^2 + \frac{1}{2}m_2 u_2^2 = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$;

Substitute m_1 = 4kg, m_2 = 1kg, u_1 = 1ms^{-1} and u_2 = –2ms^{-1} into the above equations, we have v_1 = –0.2 ms^{-1}, v_2 = 2.8 ms^{-1}.

(2) By the conservation of momentum $m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$,

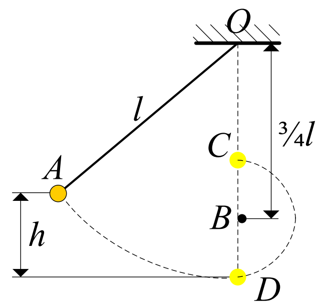
By the conservation of mechanical energy $\frac{1}{2}m_1 u_1^2 + \frac{1}{2}m_2 u_2^2 = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$ +loss in mechanical energy (Internal energy, sound or light),

Thus $(1-25\%)\cdot\left(\frac{1}{2}m_1 u_1^2 + \frac{1}{2}m_2 u_2^2\right) = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$;

Substitute m_1 = 4kg, m_2 = 1kg, u_1 = 1ms^{-1} and u_2 = –2ms^{-1}, we have

v_1 = –0.110 ms^{-1}, v_2 = 2.44 ms^{-1}.

<u>Example 2</u>



A bob A is hung by a light string of length l at O. There is a nail which is at $\frac{3}{4}l$ below point O. The bob is now displaced to a initial height h and released. The string is hampered by the nail B when it passes through the vertical position. Only the portion of the string below the nail can continue to swing to the right. Assume the string is stretched during the circular motion of the bob.

(1) If the initial height h is equal to l (the string is horizontal), find (i) the velocity v and (ii) the tension T of the string when the bob passes the point C which is vertically above the nail.

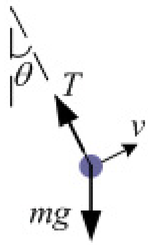
(2) If the initial height h is smaller than l ($h < l$), find (i) the minimum velocity and (ii) the minimum initial height h of the bob so that it can just pass throught the point C which is vertically above the nail B.

(The result should be expressed by l and g)

<u>Solution</u>

(1)(i) Take the point C as zero potential energy. The mechanical energy of the ball is conserved from A to C, i.e. $\frac{1}{2}mv^2 = mg\frac{l}{2}$.

$\therefore$ velocity of the pendulum bob $v = \underline{\sqrt{gl}}$ 。



(ii) As shown in the diagram, centripetal force is $m\frac{v^2}{R} = T - mg\cos\theta$.

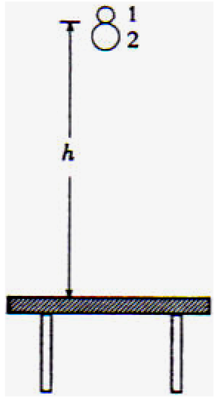
At point C, $\theta = 180^0$， $R = \frac{l}{4}$.

$\therefore$ Tension of the string $T = m\frac{gl}{l/4} + mg\cos 180^0 = \underline{3\ mg}$.

(2) (i)Centripetal force $m\frac{v^2}{R} = T - mg\cos\theta$. When T=0, the bob can just pass the point C. At this moment, $m\frac{v^2}{l/4} = -mg\cos 180^0$. That is $v^2 = \frac{gl}{4}$ ， $\therefore\ v = \underline{\frac{\sqrt{gl}}{2}}$.

(ii) Take the point C as zero potential energy and the mechanical energy is conserved from A to C. Then, we have

$$mg\left(h - \frac{l}{2}\right) = \frac{1}{2}mv^2\text{, i.e. } h = \frac{l}{2} + \frac{v^2}{2g} = \frac{l}{2} + \frac{1}{2g}\cdot\frac{gl}{4} = \underline{\frac{5}{8}}l.$$



Example 3 (HKPhO 2004 open question #4)

There are two elastic balls that nearly touch each other. The line joining their centre of mass is vertical. They fall at a height H_0 freely. The masses of the ball at the bottom and at the top are M=10m and m respectively. Assume the collision between the ball and the ground and the collision between the balls are elastic. Find the maximum height of the ball at the top after rebounce.

Solution

When they reach the ground, the velocity is $v=\sqrt{2gH_0}$ downward (i)

The ball at the bottom rebounds with the same speed and then collide with the ball at the top.

By conservation of momentum $-m_1v+m_2v=m_1v_1+m_2v_2$， (ii)

By conservation of energy $\frac{1}{2}m_1v^2+\frac{1}{2}m_2v^2=\frac{1}{2}m_1{v_1}^2+\frac{1}{2}m_2{v_2}^2$， (iii)

From (ii) and (iii)，we have $v_1=\frac{3m_2-m_1}{m_1+m_2}v,\quad v_2=\frac{m_2-3m_1}{m_1+m_2}v$ (iv)

Substitute M=10m into (iv), we have $v_1=\frac{29}{11}v,\quad v_2=\frac{7}{11}v$

The maximum height of the ball at the top $H=\frac{{v_1}^2}{2g}=\left(\frac{29}{11}\right)^2\frac{v^2}{2g}$

From (i), we have $H_0=\frac{v^2}{2g}$,

Then, $H=\left(\frac{29}{11}\right)^2H_0\approx\underline{\underline{6.95H_0}}$.

Discussion

Take $k=\frac{m_2}{m_1}$，expression (iv) changes to $v_1=\frac{3k-1}{k+1}v=\frac{3-\frac{1}{k}}{1+\frac{1}{k}}v,\quad v_2=\frac{1-3k}{k+1}v=\frac{1-\frac{3}{k}}{1+\frac{1}{k}}v$,

Then $3\le k<\infty,\quad 2v\le v_1<3v, 0\le v_2<2v,\quad 4H_0\le H<9H_0$ is satisfied。

Example 4 (The 1st Pan-Pearl River Delta Region Physics Olympiad Contest Question 2)

Two rubber balls of mass m are moving on the same straight line after collision.

(a) They have a head-on collision with the same speed v and lose 36% of the kinetic energy. Find the rebounding speed of the balls.

(b) A rubber ball with speed v collides with a stationary rubber ball. Find, after collision,

(i) the velocity of the balls and (ii) the loss in kinetitc energy.

Points to note : The 36% loss of kinetic energy is only for the case of head-on collision with the same speed.

Solution

(a) Let the final velocity are u_1 and u_2 :

By conservation of momentumk,

$$mv + (-mv) = mu_1 + mu_2$$

$$\therefore \quad u_1 = -u_2 = u$$

By conservation of energy,

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}m(-v)^2 = \tfrac{1}{2}mu^2 + \tfrac{1}{2}m(-u)^2 + \text{energy loss } (36\%)$$

$$(1-36\%)\left(\tfrac{1}{2}mv^2 + \tfrac{1}{2}m(-v)^2\right) = \tfrac{1}{2}mu^2 + \tfrac{1}{2}m(-u)^2$$

$\therefore$ The rebound speed is $u = 0.8\ v$ 。

(b) (i) When a rubber ball collides with a stationary rubber ball with speed v, the centre of mass of the system is moving with speed $0.5v$. As there is no external force and all the forces during collision collision are internal forces, the speed of centre of mass is unchanged.

Take the centre as the reference frame. The centre of mass is at rest and the balls are moving with $v - 0.5v = 0.5v$ and $0 - 0.5v = -0.5v$ respectively. The system loses 36% of kinetic energy.

From part (a), the rebounding speed of the balls are $\pm 0.4\ v$ according to the frame of centre of mass. Thus, the speeds of them are $= 0.5\ v \pm 0.4\ v = 0.9\ v$ 及 $0.1\ v$.

(ii) Initial kinetic energy of the system = $\dfrac{1}{2}mv^2$,

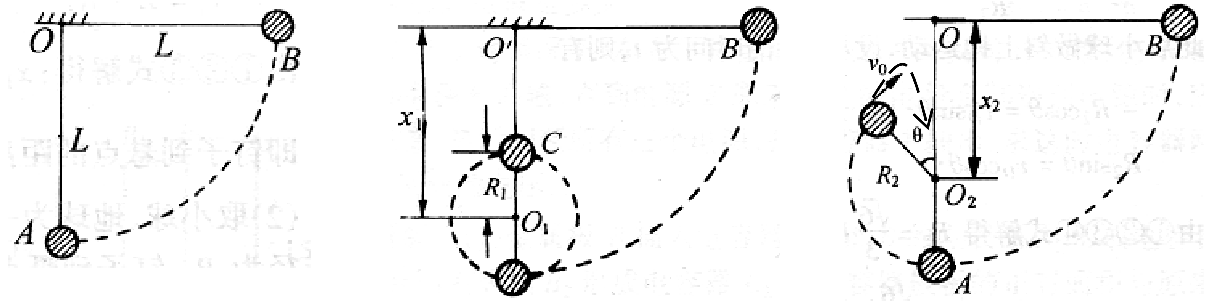
Kinetic energy after collision = $\tfrac{1}{2}m(0.9v)^2 + \tfrac{1}{2}m(0.1v)^2 = 0.82 \times \tfrac{1}{2}mv^2 = (1-18\%) \times$ kinetic energy before collision

$\therefore$ 18% of kinetic energy is lost during collision.

<u>Example 5</u> (The 1st Pan-Pearl River Delta Region Physics Olympiad Contest Question 5)

As shown in the diagram, a bob of mass m is attached to the end of a light thread of length L which is fixed at the point O on the ceiling. Point A is the position of the ball at equilibrium. The bob is lifted from A to B at which the string is horizontal. A nail is then hinged at a point on the line joing OA.

The bob is then released. When it passes throught its equilibrium position, the string is blocked by the nail and only the portion of the string below the nail can swing to the left. Assume the string is stretched during the circular motion of the bob.



(a) (b)

In the following situations, what is the distance (x_1 and x_2) of the nail from the point O ?

(a) The pendulum bob can pass through the point C which is vertically above the nail.

(b) The pendulum bob performs projectile motion at the point D and hits the nail O_2.

Solution

(a) Take the point C as zero potential energy. The mechanical is conserved when the bob swings from B to C. We have

$$mg(L-2R_1)=\frac{1}{2}mv_C^{\,2},\ v_C^{\,2}=2g(L-2R_1);\quad (1)$$

centripetal force : $m\dfrac{v^2}{R}=T-mg\cos\theta$,

when tension $T=m\dfrac{v^2}{R}+mg\cos\theta\geq 0$, the bob can pass through the point C. At the moment, θ $=180^0$. Then, we have $v_C^{\,2}-gR_1\geq 0$, (2)

Solving (1) and (2), we have $R_1\leq\dfrac{2}{5}L$,

$$\therefore\ x_1=L-R_1\geq\underline{\underline{\frac{3}{5}L}}$$

(b) Take point D as the zero potential energy. When the bob swings from B to D, the mechanical energy is conserved.

$$mg(L-R_2-R_2\cos\theta)=\frac{1}{2}mv_D^{\,2},$$

ie. $$v_D^{\,2}=2g(L-R_2-R_2\cos\theta);\quad (1)$$

centripetal force: $m\dfrac{v_D^{\,2}}{R_2}=T+mg\cos\theta$, 其中 $T=0$ (2)

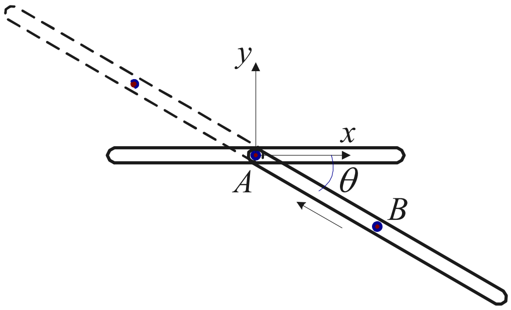
The bob is projected at an angle at D and hit the nail O_2. The time is t.

$$-R_2\cos\theta=v_D\sin\theta\cdot t-\frac{1}{2}gt^2,\quad (3)$$

$$R_2\sin\theta=v_D\cos\theta\cdot t\quad (4)$$

From (1), (2), (3) and (4), we have $R_2=\sqrt{\dfrac{2}{3}}L$, $x_2=L-R_2=\underline{\underline{\left(1-\sqrt{\dfrac{2}{3}}\right)L}}$.

<u>Example 6</u>(The 1st Pan-Pearl River Delta Region Physics Olympiad Contest Question 1)
There are two uniform worms of length $2l$ and masses m_A and m_B on a smooth plane. Their initial poisitons are shown as the solid lines in the diagram. The centre of mass of worm A is at the coordinate (0,0) of the x-y coordinate system.



(a) Using l， θ、m_A and m_B , express the coordinates of the centre of masses of the worm B and the system of $A+B$.
(b) The worm B then scrambles over the worm A slowly. The angle between the worms is kept constant at θ. Determine the coordinates of the centre of mass of the worms when the worm B has just passed the worm A.

<u>Solution</u>
(a) The coordinates of centre of mass of the worms before :

$$A(0，0)，\quad B((l\cos\theta,\ -l\sin\theta)，$$

The centre of mass of the system is $C\left(\frac{m_B l\cos\theta}{M},\ -\frac{m_B l\sin\theta}{M}\right)$, in which $M = m_A + m_B$。

(b) When the worm B has just scambled over the worm A, the coordinates of their centre of mass are :

$$A(X，Y)，\quad B(X - l\cos\theta,\ Y + l\sin\theta)，$$

Then, the centre of mass of the system is $\left(\frac{m_A X + m_B(X - l\cos\theta)}{M},\ \frac{m_A Y + m_B(Y + l\sin\theta)}{M}\right)$.

By the conservation law of centre of mass, the position of the centre of mass is unchanged as there is no horizontal external force.

$$\left(\frac{m_B l\cos\theta}{M},\ -\frac{m_B l\sin\theta}{M}\right) = \left(\frac{m_A X + m_B(X - l\cos\theta)}{M},\ \frac{m_A Y + m_B(Y + l\sin\theta)}{M}\right)$$

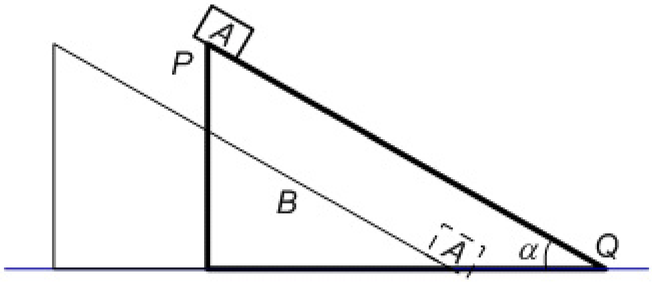
$\therefore$ Thus, the coordinates of A and B is

$$A(X, Y) = A\left(\frac{2m_B}{M} l\cos\theta, -\frac{2m_B}{M} l\sin\theta\right)$$

$$B(X - l\cos\theta,\ Y + l\sin\theta) = B\left(-\frac{m_A - m_B}{M} l\cos\theta, \frac{m_A - m_B}{M} l\sin\theta\right)$$

<u>Example 7</u> (The 2nd Pan-Pearl River Delta Region Physics Olympiad Contest Question 5)

Object B of mass M is placed on a smooth horizontal surface. Object A of mass m slides from P to Q along a smooth inclined surface of B. Let the length of PQ is L. When A (can be seen as a point mass) is slided from P to Q, find (a) the horizontal displacement s, (b) acceleration a of B and (c) the time taken t.

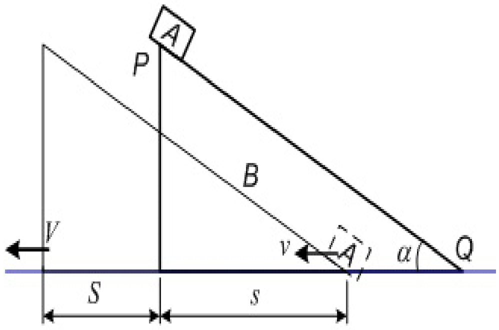


<u>Solution</u>

(a) Take $3b = l\cos\alpha$ and point P as the origin of the horizontal axis.

When the object A is at the initial position P, the cetnres of mass are $A(0)$和 $B(0)$.

The centre of mass of the system is $C_P\left(\dfrac{M}{m+M}b\right)$.



When the object A reaches the point Q, the centres of mass are $A(x_A)$ and $B(x_B)$.

The centre of mass of the system is $C_Q\left(\dfrac{mx_A + Mx_B}{m+M}\right)$

and the separation the centres of mass of the objects is $x_A - x_B = 2b$.

By the conservation law of centre of mass, the position of the centre of mass is unchanged as there is no horizontal external force.

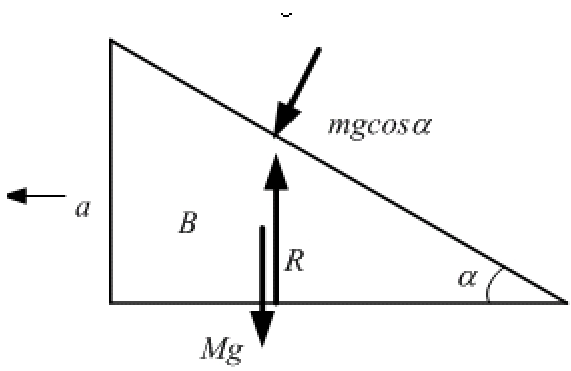
$$C_P\left(\frac{M}{m+M}b\right) = C_Q\left(\frac{mx_A + Mx_B}{m+M}\right), \quad \text{i.e. } \frac{M}{m+M}b = \frac{mx_A + Mx_B}{m+M}\text{。}$$

$\therefore$ The coordinates of the centres of mass are $x_A = \dfrac{3M}{m+M}b$ and $x_B = -\dfrac{2m-M}{m+M}b$.

(b)

$\sum F_x = ma$, $(mg\cos\alpha)\sin\alpha = Ma$

$$a = \frac{m}{M}g\sin\alpha\cos\alpha$$



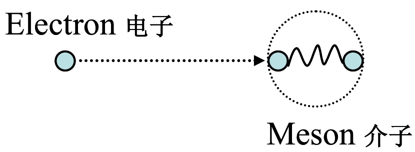
The displacement of B :

$$s = b - x_B = b - \frac{2m-M}{m+M}b = \frac{3m}{m+M}b = \frac{1}{2}at^2,$$

$$\frac{m}{m+M}l\cos\alpha = \frac{1}{2}\times\frac{m}{M}g\sin\alpha\cos\alpha\times t^2$$

$$t^2 = \frac{M}{m+M} \cdot \frac{2l}{g\sin\alpha}, \quad \therefore t = \sqrt{\frac{M}{m+M} \cdot \frac{2l}{g\sin\alpha}}$$

Example 8 (The 2nd Pan-Pearl River Delta Region Physics Olympiad Contest Question 6)
Meson is composed of two quarks in which the interaction between the quarks is complicated. Research of meson can be done by colliding the meson of high energy electrons inelastically. As the collision is too complicated to be analyzed, scientists invented a simplified model called "partial particle" to grasp the main content during collision. In the model, the electron collides with part of the meason (e.g. one of the quarks) elastically. Then the energy and momentum are transferred to the other quark and thus the whole meson during interaction .
This simplified model is described by the following :



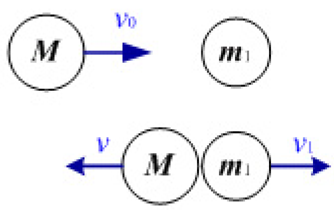
An electron of mass M and energy E collide with a quark of mass m_1 which is in a meson. The other quark in the meson has a mass m_2. The quarks are connected by a massless spring of natural length L which is at equilibrium before collision. All movements are on a straight line and neglect the effect of relativity. Find, after collision,
(a) the energy gain of the quark m_1;
(b) the kinetic energy of the meson as a whole system, and
(c) the internal energy of the meson which is expressed by the oscillation of the spring.

Solution

(a) The velocity of the electron is $v_0 = \sqrt{\frac{2E}{M}}$. The velocity of the electron and the quark after collision are v and v_1.

$$Mv_0 = Mv + m_1 v_1 \text{ ,} \qquad \text{(i)}$$

$$E = \frac{1}{2}Mv^2 + \frac{1}{2}m_1 v_1{}^2 \text{ ,} \qquad \text{(ii)}$$



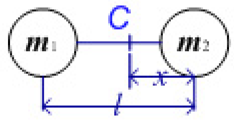
From (i), we ahve $v = v_0 - \frac{m_1}{M} v_1$,

substitute into (ii), we have $v_1 = \frac{2v_0}{1+\frac{m_1}{M}} = \frac{2M}{M+m_1} v_0$.

The kinetic energy gain of the quark m_1 is

$$E_m = \frac{1}{2}m_1 v_1{}^2 = \frac{1}{2}m_1\left(\frac{2M}{M+m_1}v_0\right)^2 = \frac{4Mm_1}{(M+m_1)^2}E \text{ 。}$$

(b) The position and velocity of the centre of mass C :



$$m_1 l = (m_1 + m_2)x \text{ ，} \quad \frac{x}{l} = \frac{m_1}{m_1 + m_2} = \frac{v_C}{v_1} \text{ ，}$$

$$v_C = \frac{m_1}{m_1 + m_2} v_1 = \frac{m_1}{m_1 + m_2} \frac{2M}{M + m_1} v_0 = \frac{2Mm_1}{(M + m_1)(m_1 + m_2)} v_0$$

The kinetic energy of the centre of mass :

$$E_k = \frac{1}{2}(m_1 + m_2) v_C{}^2 = \frac{1}{2}(m_1 + m_2)\left(\frac{2Mm_1}{(M + m_1)(m_1 + m_2)} v_0\right)^2 = \frac{4Mm_1{}^2}{(M + m_1)^2 (m_1 + m_2)} E$$

$E_k = \dfrac{4Mm_1{}^2}{(M + m_1)^2 (m_1 + m_2)} E$, which is the kinetic energy of meson.

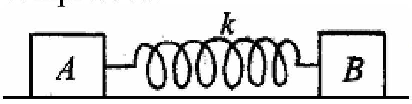
(c) The kinetic energy gained by the quark m_1 after collision is the energy of the meson
= the kinetic energy of the meson as a system + internal energy in the oscillation of the spring
∴ The internal energy of the meson which is expressed by the oscillation of the spring is

$$E_i = E_m - E_k = \frac{4Mm_1}{(M + m_1)^2} E - \frac{4Mm_1{}^2}{(M + m_1)^2 (m_1 + m_2)} E$$

$$E_i = \frac{4Mm_1 m_2}{(M + m_1)^2 (m_1 + m_2)} E$$

Example 9 (The 2nd Pan-Pearl River Delta Region Physics Olympiad Contest Question 7)

A massless spring of spring constant k is connecting two objects A and B of mass m (similar to the structure of a meson in example 8). They are placed on a smooth horizontal table. The object A is moving with velocity v_0 towards the object B which is at rest initially. The spring is being compressed.



(a) Describe the motion of the centre of mass of the system.
(b) Determine the period of the oscillation.
(c) Determineo the maximum compression of the spring.
(d) From the beginning to the maximum compression of the spring, what are the displacements of the the object A and B relative to the table ?

Solution
(1) As there is no external horizontal force, the centre of mass moves with uniform speed. The speed of the centre of mass v_c :

$$mv_0 = 2mv_c, \quad v_c = \frac{v_0}{2}$$

(2) Take the centre of mass as the reference frame. The centre of mass is stationary and each of the object performs simple harmonic motion under the influence of half of the spring. The effective force constant is k', where $k' = 2k$ ， $v_{\max} = \dfrac{v_0}{2}$.

The period of the simple harmonic motion is $\underline{\underline{T = 2\pi\sqrt{\dfrac{m}{2k}}}}$ and

$m_1 v_0 = (m_1 + m_2)v_c$, $v_c = \dfrac{m_1}{m_1 + m_2} v_0$.

$T = 2\pi\sqrt{\dfrac{m_1 m_2}{k(m_1 + m_2)}}$, force constant $k_1 = \dfrac{k(m_1 + m_2)}{m_2}$, $k_2 = \dfrac{k(m_1 + m_2)}{m_1}$.

(3) (When the velocity of the springs are the same (i.e. v), the spring reaches its maximum compression x. By conservation of momentum, we have $mv_0 = 2mv$. By conservation of energy, we have $\dfrac{1}{2}mv_0{}^2 = \dfrac{1}{2}mv^2 \times 2 + \dfrac{1}{2}kx^2$. Thus, the maximum compression of the spring is $x = v_0\sqrt{\dfrac{m}{2k}}$.)

During the process is compressed to its maximum compression, the objects are moved from equilibrium to the maximum compression of the spring according to the frame of the centre of mass. Thus, by conservation of mechanical energy :

$$\frac{1}{2}mv_{\max}{}^2 = \frac{1}{2}k'x'^2, \frac{1}{2}m\left(\frac{v_0}{2}\right)^2 = \frac{1}{2}\cdot 2k \cdot x'^2, x' = \frac{v_0}{2}\sqrt{\frac{m}{2k}}$$

From this, the maximum compression is $x = v_0\sqrt{\dfrac{m}{2k}}$.

(4) The objects are moved from equilibrium to the maximum compression of the spring according to the frame of the centre of mass. The time needed is a quarter of one period. Thus, $t = \dfrac{T}{4} = \dfrac{\pi}{2}\sqrt{\dfrac{m}{2k}}$.

Within this duration, the displacement of the centre of mass is $x_c = v_c t = v_0 \dfrac{\pi}{4}\sqrt{\dfrac{m}{2k}}$.

According to the frame of centre of mass, the relative displacement of the objects A and B is $x_c{}' = \pm x' = \pm\dfrac{v_0}{2}\sqrt{\dfrac{m}{2k}}$.

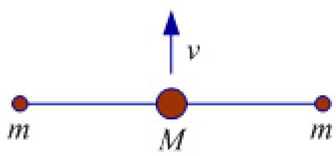
By the principle of relative motion, the displacement of the objects A and B is

$$x_{A,B} = x_c + x_c{}' = \frac{v_0}{2}\left(\frac{\pi}{2} - 1\right)\sqrt{\frac{m}{2k}}.$$

Examples on inertia force

Q4 of HKPhO 2005

As shown, a large ball of mass M is connected on each end by a weightless thread of length l to a small ball of mass m. Initially the three balls are along the straight line on a smooth surface. The large ball is suddenly given an initial velocity v in the direction perpendicular to the line. Find



(a) The tension in the thread at the moment the large ball gets the impact;
(b) The tension in the thread at the moment the two small balls meet.

Solution

(a) Consider we observe the motion in the reference frame of mass M, the two small mass m will seen to be performing circular motion with initial velocity $-v$. The acceleration of M, by symmetry of the forces acting upon it, will be along $-v$ and perpendicular to the acceleration of the small masses.

So, we have

$$T = \frac{mv^2}{l}$$

(b) The small masses are moving around the big mass M in circular motion, but the velocity is not constant. The big mass itself is accelerating with unknown acceleration a_M.

$$2T_2 = Ma_M \quad \Rightarrow \quad a_M = \frac{2T_2}{M} \qquad (1)$$

Taking into account the inertia force, we have：

$$T_2 + ma_M = m\frac{{v_x}^2}{l} \qquad (2),$$

where is the velocity of small mass relative to the big mass in the horizontal direction. From (1) and (2) one gets

$$T_2 = \frac{Mmv_x^2}{(M+2m)l} \qquad (3).$$

To find v_x^2 we need a second equation, that is energy conservation. The initial energy is obviously $\frac{1}{2}Mv^2$. The final energy is a bit more complicated. Let the center of mass velocity along the vertical direction as v_c, then all three balls have the same velocity in that direction. By momentum conservation

$$Mv = (M+2m)v_c \quad (4).$$

The velocity of the small balls also has a horizontal component v_x. According to conservation of energy, we then have

$$2\frac{1}{2}m(v_x^2 + v_c^2) + \frac{1}{2}Mv_c^2 = \frac{1}{2}Mv^2 \qquad (5)$$

Using (4) $\Rightarrow mv_x^2 = \frac{1}{2}Mv^2 - \frac{1}{2}(M+2m)\left(\frac{M}{M+2m}v\right)^2$

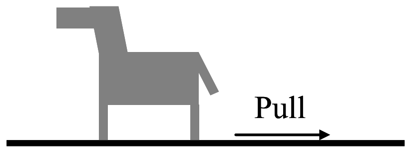
$$\Rightarrow mv_x^2 = \frac{1}{2}Mv^2\left(\frac{2m}{M+2m}\right)$$

Finally one gets

$$T_1 = \frac{M^2mv^2}{(M+2m)^2 l}$$ ans.

Inertia force can be regarded as acting on the center of mass if the object is of finite size and its balance by all torques must be considered. In other word, if the center of mass is chosen as the pivotal point, then the torque due to inertia force and gravity is zero.

Q5 of HKPhO 2005



A wooden toy horse rests on a tablecloth on a table, with its front legs 0.3 m from the cloth edge. It weighs 100 grams and its center of mass is 0.05 m from the front legs and 0.05 m above ground. The distance between the front and back legs is 0.15 m. The tablecloth is suddenly yanked horizontally with constant acceleration of 9.0 m/s^2 relative to the table. The friction coefficient between the cloth and the horse is μ = 0.75. Find

(a) the acceleration of the horse relative to the table;
(b) the force on each leg of the horse by the tablecloth;
(c) the velocity and the distance the horse has traveled relative to table when the edge of the tablecloth reaches the front legs.
(d) If the height of the center of mass could be adjusted, find the value above which the horse would tip off.

Solution

(a) Let the acceleration of horse relative to the table be a_{ht},

$$ma_{ht} = \mu mg$$
$$a_{ht} = \mu g = 0.75 \times 9.8ms^{-2} = 7.35ms^{-2}$$

(b) Consider the net moment acting on the toy horse should be zero, we have

$$N_1 + N_2 = mg$$
$$N_1 r_1 = N_2 r_2 + mg\mu h$$
$$\Rightarrow (mg - N_2) r_1 = N_2 r_2 + mg\mu h$$
$$\Rightarrow N_2 = mg \frac{r_1 - \mu h}{r_1 + r_2}$$
$$N_2 = (0.10kg)(9.8ms^{-2}) \frac{0.05m - 0.75 \times 0.05m}{0.15m}$$
$$= 8.16 \times 10^{-2} N$$

(c) Let the acceleration of horse relative to the tablecloth be a_{hc},

$$a_{hc} = a - a_{ht} = (9.0 - 7.35)ms^{-2} = 1.65ms^{-2}$$

The time required for the horse reaches the edge of tablecloth is

$$t = \sqrt{\frac{2s}{a_{hc}}} = \sqrt{\frac{2(0.3m)}{1.65ms^{-2}}} = 0.603s$$

The velocity of horse relative to table at time t is

$$v_{ht} = a_{ht} t = 4.432ms^{-1}$$

The displacement on the table is

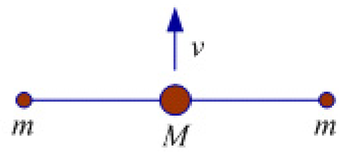
$$s' = \frac{1}{2} a_{ht} t^2 = \frac{1}{2}(7.35ms^{-1})(0.603s)^2 = 1.336m$$

(d) When the horse is fallen, $N_2 = 0$. It implies that

$$r_1 - \mu h \geq 0$$
$$h \leq \frac{r_1}{\mu} = \frac{0.05m}{0.75} = 6.67 \times 10^{-2} m$$

IPhO 2006 HK team Selection Test-1

As shown, a large ball of mass M is connected on each end by a weightless thread of length l to a small ball of mass m. Initially the three balls are along the straight line on a smooth surface. The large ball is suddenly given an initial velocity v in the direction perpendicular to the line. The subsequent collisions between the two small balls are elastic. Using rigorous derivations, find whether the motion of the large ball relative to the center of mass of the three-ball system is simple harmonic oscillation.

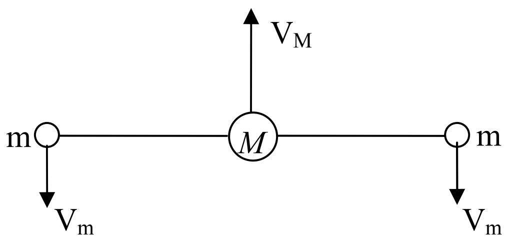


Solution

The velocity of the centre of mass is $\dfrac{MV}{M+m}$, which remains the same all the time.

Take the cetnre of mass as the reference frame :

Initially, the speed of the balls are

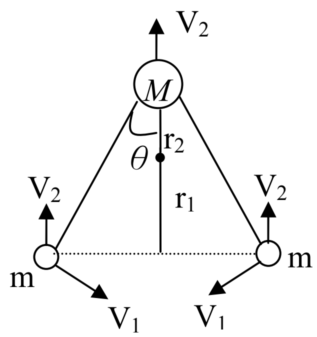


$$V_M = \frac{2mV}{M+2m}$$

$$V_m = \frac{-MV}{M+2m}$$

The kinetic energy is $E = \dfrac{1}{2}MV_M{}^2 + \dfrac{1}{2}\cdot 2m\cdot V_m{}^2 = \dfrac{Mm}{M+2m}V^2$

In the intermediant stage, the velocities of the small balls relative to the centre of mass afterwards are:



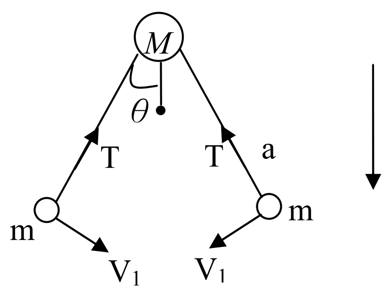
$$(M+2m)V_2 = 2mV_1\sin\theta$$

$$\frac{1}{2}MV_2^2 + \frac{1}{2}\cdot 2m\left[(V_2 - V_1\sin\theta)^2 + V_1^2\cos^2\theta\right] = E$$

$$\Rightarrow V_1^2 = \frac{MV^2}{M+2m\cos^2\theta}$$

$$V_2^2 = \frac{4m^2}{(M+2m)^2}\cdot\frac{MV^2}{M+2m\cos^2\theta}\cdot\sin^2\theta$$

M is at a distance r_2 from the center of mass. If M is in simple harmonic motion, the force on it should be proportional to r_2. Take the reference frame which is moving with the big ball M. The force f acting on M :



$$f = 2T\cos\theta = Ma$$

$$T = \frac{mV_1^2}{R} - ma\cos\theta$$

$$\Rightarrow T = \frac{mV^2}{R(1+2\cos^2\theta)}$$

$$a = \frac{2mV^2\cos\theta}{MR(1+2\cos^2\theta)}$$

Take the centre of mass as the reference frame,
$r_2 = \dfrac{2mR\cos\theta}{M+2m}$, so it is not linearly proportional to *a*.
As the relationship is non-linear, the motion is not simple harmonic.

6.6 Appendix: The relationship among Newton's law of motion, conservation of kinetic energy and conserfvation of momentum

6.6.1 Conserfvation of Momentum

From $\vec{F} = m\vec{a} = m\dfrac{d\vec{v}}{dt} = \dfrac{d(m\vec{v})}{dt} = \dfrac{d\vec{p}}{dt}$, if $\vec{F}$ is unchanged, we have $\vec{F}\Delta t = \vec{p}_2 - \vec{p}_1$. That is, impulse = change of momentum. If $\vec{F}$ is varying, then $\vec{F}dt = d\vec{p}$ and the change in momentum $d\vec{p}$ is equal to the impulse $\vec{F}dt$ in a very short time *dt*.

Consider a two bodies system, the initial momentum of the particles are $\vec{p}_A$ and $\vec{p}_B$. The initial total momentum of the system is $\vec{p}_A + \vec{p}_B$. The force acting on the particle *A* by *B* is $\vec{F}$. By the Newton's third law of motion, the action is equal but opposite to the reaction force. Thus, the force acting on B by A is $-\vec{F}$. After a short time *dt*, the momenta of particle *A* and B become $\vec{p}_A - \vec{F}dt$ and $\vec{p}_B + \vec{F}dt$ respectively. The total momentum is conserved. The total momentum is conserved in a many-bodies system and can be derived in a similar manner.

6.6.2 Conserfvation of Energy

Similarly, if $\vec{F}$ is unchanged, the particle moves with uniform acceleration. After a displacement *S*, we have $2aS = v_2^2 - v_1^2$, in which *a* is acceleration, v_1 and v_2 are the initial and final velocity respectively. Also, the work done is

$$W = F \cdot S = maS = \frac{1}{2}m(v_2^2 - v_1^2) = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 \circ$$

Therefore, the work done by the external force is equal to the change in kinetic energy of the particle.

With further development of Physics, more and more forms of energy are discovered. The application of the laws of conservation of energy and momentum are widened in different areas, and are eventually elevated to the status of the fundamental laws in nature. The Newtonian mechanics was replaced by new theory. The concept of force is faded out in the new theories of 'field' and 'interaction'. According to Newtonian mechanics, force is the cause of the laws of conservation of energy and momentum. Actually, the laws of conservation of energy and momentum are fundamental laws in nature and the Newtonian mechanics is a particular case under certain criteria.

Chapter 7 Simple Harmonic Motion

7.1 Definition: Simple Harmonic Motion (SHM) occurs when the net force acting on an object is proportional to its displacement and in opposite direction: i.e.

$$F = -kx = ma$$

7.2 Characteristics:

(a) The oscillation is *sinusoidal and periodic*;

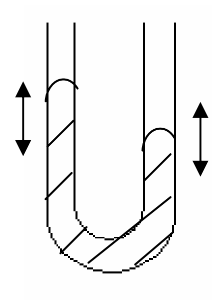
(b) The acceleration is always ***directed toward to a fixed point*** – equilibrium position; (restoring forces that "obey" Hooke's Law;

(c) The acceleration is proportional to its displacement from that equilibrium position; and

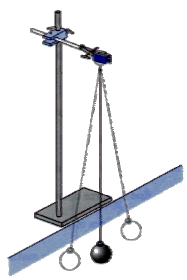
(d) The period is ***independent*** of the amplitude for a given system.

7.3 Daily Examples of Harmonic Motion:

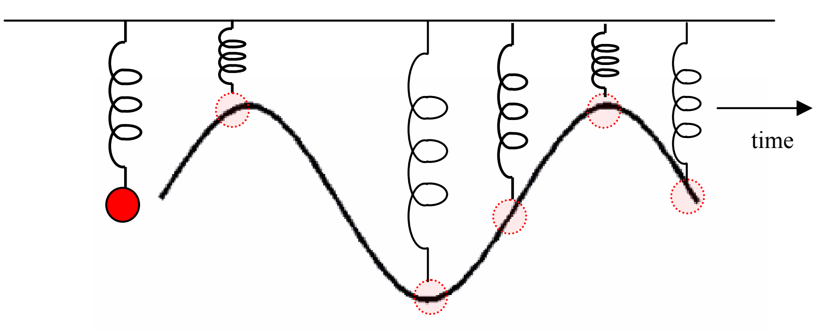
(a) Mercury oscillating in the U - tube



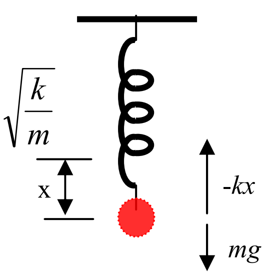
(b) A simple pendulum



(c) Mass oscillating on the end of a fastened spring



$$\omega = \sqrt{\frac{k}{m}}$$



Hooke's Law:
$\boldsymbol{F_{spring} = -kx}$

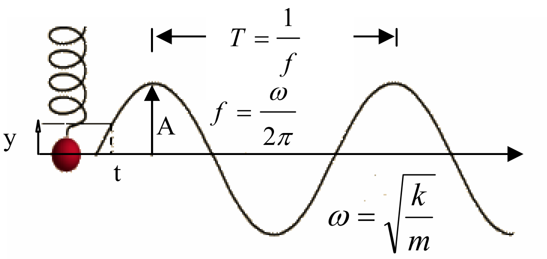
(d) Amusement park rides

7.4 Useful Terms:

(a) Period (T): the time for one complete cycle of the oscillation.

(b) Frequency (f): the number of cycles of oscillation per second, and note that $f = 1/T$.

(c) Amplitude (A): the maximum value of the displacement from equilibrium

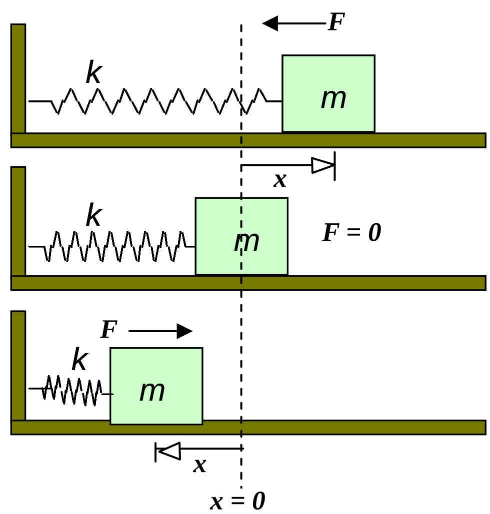


(d) Isochronous oscillation takes the same time for each complete oscillation whatever its amplitude.

Note: Not all repetitive motions over the same path can be classified as a *SHM*. For example, consider a ball being tossed back and forth between a parent and child. The ball moves repetitively, but the motion is not *SHM*. Unless the force acting on an object along the direction of motion has the form of equation $F = -kx$.

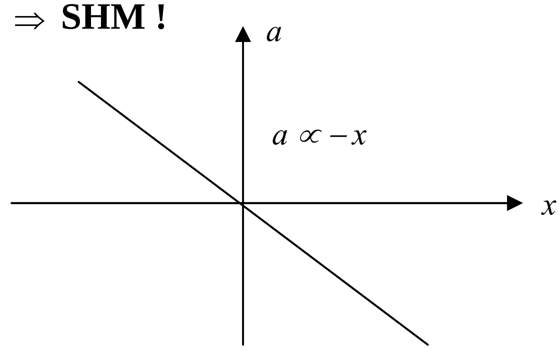
7.5 Qualitative description about SHM-mass on a spring:

A typical example of SHM (mass-spring system) is shown at right. Assume no friction. The point $x = 0$ is the equilibrium point of the mass. In this position, there is no spring restoring force acting on the mass. When the mass is displaced a distance x from its equilibrium position, the spring produces a restoring force given by Hooke's law, $F = -kx$, where k is the force constant of the spring. The minus sign means that F is to the left when the displacement x is positive, whereas F is to the right when x is negative. In other words, the direction of the force is always towards the equilibrium position.



□ $F = ma = -kx \Rightarrow a = -\frac{k}{m}x$

$$a = -\text{(constant)} \times x \quad \Rightarrow \text{SHM !}$$



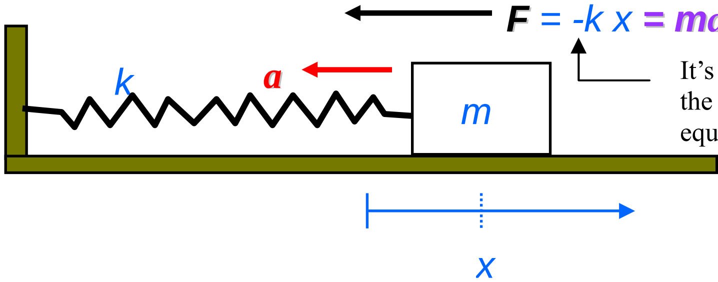
A graph of acceleration against the displacement of an object

7.6 Studying a Simple Harmonic Motion

Force proportional to displacement 'Linearity'

$F = -k\,x = ma$

It's negative, because force resists against the change of length, directed toward the equilibrium position.

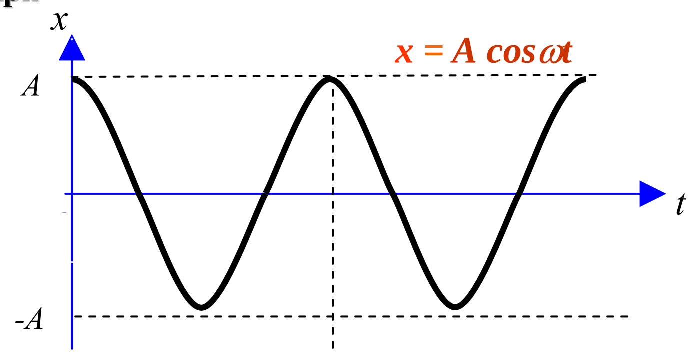


Position: $x(t) = A\cos(\omega t)$
Velocity: $v(t) = -\,\omega A\sin(\omega t)$
Acceleration: $a(t) = -\,\omega^2 A\cos(\omega t)$

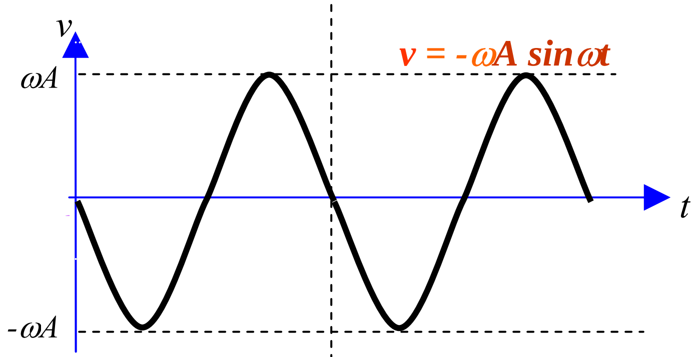
by taking derivatives, since we get

$$\boxed{v(t) = \frac{dx}{dt};\ a(t) = \frac{dv}{dt}}$$

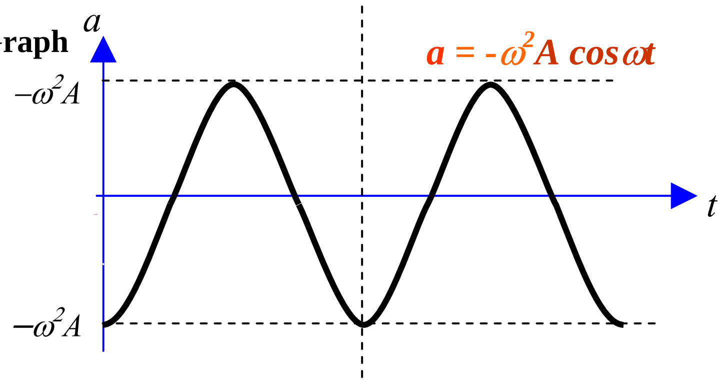
Displacement-Time Graph



Velocity-Time Graph



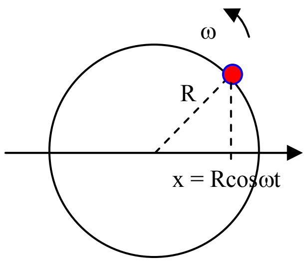
Acceleration –Time Graph

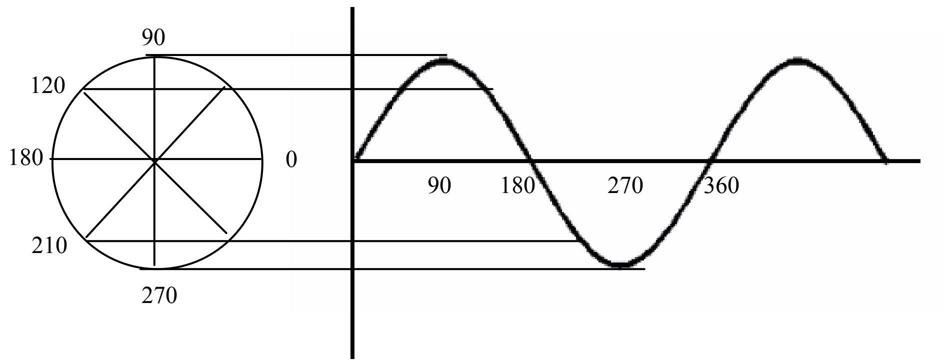


Note:
1. <u>All three graphs are sinusoidal</u>
2. The velocity/time, and acceleration/time graphs are 90^o and 180^o out of phase, respectively, with the displacement.

Check Point 1: A base ball is dropped onto the ground floor. Over and over, it rebounds to its original height. Is the motion a simple harmonic motion? Justify your reasoning.

7.7 Comparing SHM with uniform circular motion:
An experimental setup for demonstrating the connection between *SHM* and uniform circular motion. A particle rotates on the turntable with constant angular velocity □, its shadow on the screen moves back and forth with *SHM*.





The graph of y against t is sinusoidal, with a period $T \equiv 2\pi / \omega$, whereas a, the radius of the circle, becomes the amplitude of the motion.

SHM considered mathematically
Let □ be the angular velocity and at $t = 0$.

Suppose a particle with the following motion:

$$x = A\cos\omega t; \quad v = \frac{dx}{dt} = -\omega A\sin\omega t; \quad a = \frac{d^2x}{dt^2} = -\omega^2 A\cos\omega t \qquad \ldots. (1)$$

Applying Newton's 2nd law:

$$F = -kx = ma = m\frac{d^2x}{dt^2} \Rightarrow \frac{d^2x}{dt^2} = -\left(\frac{k}{m}\right)x \qquad \ldots. (2)$$

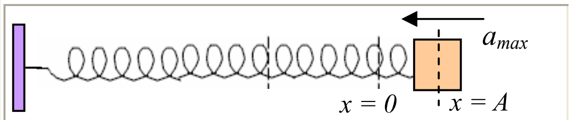
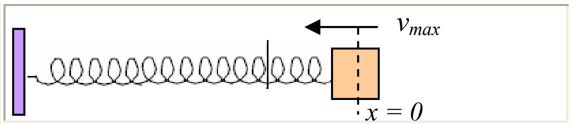
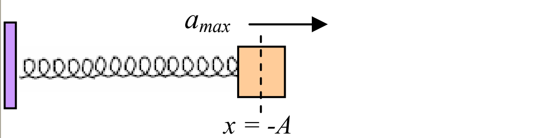
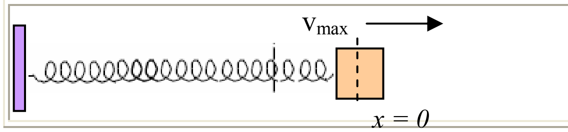
Equating (1) and (2), we have

$$-\omega^2 A\cos\omega t = -\left(\frac{k}{m}\right)A\cos\omega t \quad \Rightarrow \quad \omega = \sqrt{\frac{k}{m}}$$

Since $\boxed{T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \Rightarrow f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}}$

Note: <u>**ωt**</u> is measured in terms of ***<u>radian.</u>***

Relationship between displacement, velocity and acceleration of a SHM.

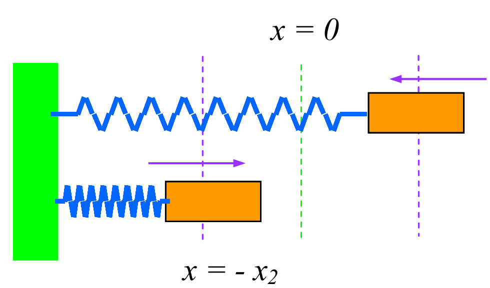
Spring and block system	displacement	velocity	acceleration	Period
	$+A$	0	$-\omega^2 A$	$t = 0$
	0	$-\omega A$	0	$t = 1/4\ T$
	$-A$	0	□$\omega^2 A$	$t = 1/2\ T$
	0	□A	0	$t = 3/4\ T$

Check Point 2: If a mass-spring system is hung vertically and set into oscillation, why does the motion eventually stop?

Example 1

Let $A = 20\ cm;\ \omega = 2\ rad/s$. (a) Find the time taken t for $x = 10\ cm$.
(b) Find the position of the mass x when $t = 0.52\ s$

Solution

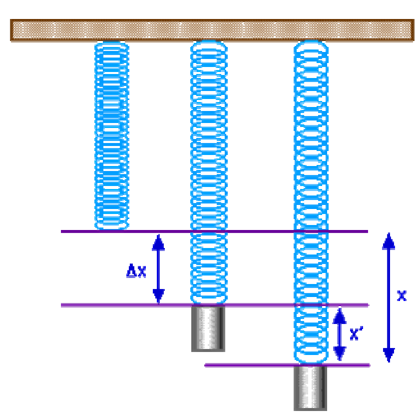


(a) By using the equation, $x = A\cos\omega t$

$$10 = 20\cos(2t)$$
$$0.5 = \cos(2t)$$
$$60^o = (2t)\ rad$$
$$1.046\ rad = 2t\ rad$$
$$t = 0.52\ s$$

(b) $x = A\cos\omega t \quad = 20\cos[2(0.52\ s)] = 20\cos[1.04rad] = 20\cos(60^o) = 10\ cm$

7.8 Mass hanging from a spring:



Three diagrams as shown at right help us to have a better understanding about the spring forces acting on an object under the influence of gravity.
The first figure shows the spring of length l, unloaded. The second figure shows the spring loaded but in its equilibrium position and the spring extended by a distance □x. The third one is a 'snapshot' at a point in the cycle of oscillation with a displacement, x, from the rest position of the 2^{nd} diagram.

Applying Hooke's law to the spring, we obtain

$$F = kx$$

Because of equilibrium, F = weight of the mass m, giving

$$kx = mg, \quad or \quad k = mg/x$$

In the 3^{rd} diagram, the force which is acting upward, is given by

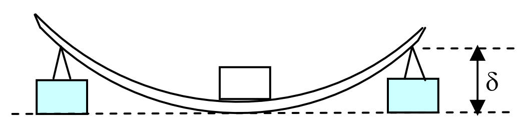
$$F = k(x + x)$$

Take x as positive when increasing in a downward direction from the equilibrium position. The net force F in the direction of increasing x is then:

$$\begin{aligned} F &= mg - k(x + x) \\ &= mg - kx - kx \\ &= -kx \end{aligned}$$

Therefore it is the result which proves the motion is *SHM*. We have a restoring force proportional to the displacement.

$$F = ma = -kx \qquad \Rightarrow a = -\frac{k}{m}x \quad or \quad a = -\omega^2 x$$



<u>Example 2</u>

Given that the static deflection of a light beam under a load is □.

Show that the period of vibration of the load is $T = 2\pi\sqrt{\dfrac{\delta}{g}}$.

Assume that the load remains in contact with the beam.

<u>Solution</u>

The motion of the beam is similar to the spring-mass system, thus we have

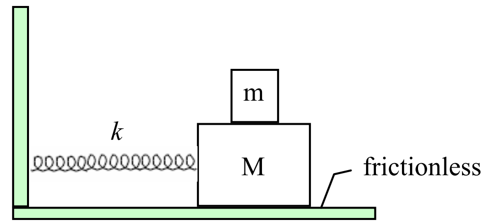
$$mg = k□ \qquad \Rightarrow k = mg/\delta$$

For simple harmonic motion, $ma + kx = 0 \qquad \Rightarrow a + \dfrac{k}{m}x = 0 \;\; ; \;\; \omega = \sqrt{\dfrac{k}{m}}$

Thus $T = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{m}{k}} = 2\pi\sqrt{\dfrac{\delta}{g}}$

Example 3

Two blocks ($m = 1.0\ kg$ and $M = 5.0\ kg$) and a massless spring ($k = 100\ N/m$), as shown in figure are arranged on a frictionless surface. Given that the coefficient of static friction between two blocks is *0.4*. What is the maximum amplitude of such *SHM* of spring-blocks system if no slipping is to occur between the blocks?



Solution

To be on the verge of slipping means that the force exerted on the smaller block (at the point of maximum acceleration) is f_{max} = □$_s mg$. The acceleration of the smaller block at that moment is a_{max} =□$^2 x_{max}$, where □ is the angular frequency ($\omega = \sqrt{k/(m+M)}$). Therefore, using Newton's second law, we have

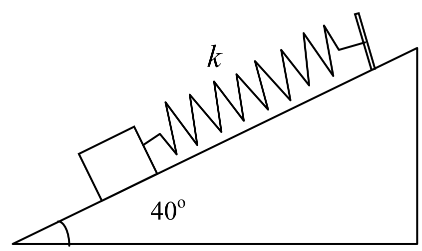
$$f_{\max} = ma_{maz} = m(\omega^2 x_{\max}) = \mu_s mg$$

$$\Rightarrow \quad (\frac{\mathrm{k}}{\mathrm{m+M}}) \cdot x_{\max} = \mu_s g,$$

which leads to $x_{max} = 0.24\ m$

Example 4

As shown in figure at right, if the block weighing *100 N* is pulled slightly down the incline and release, what is the period of the resulting oscillation? (Given $k = 120\ N/m$)



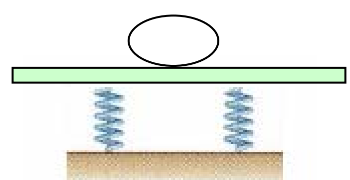
Solution

Just as with a vertical spring, the effect of gravity is simply to shift the equilibrium position; it does not change the characteristics (such as the period) of simple harmonic motion. Thus we obtain

$$\text{The period } \ T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{100/9.8}{120}} = 1.83\, s$$

Example 5

A block of mass *0.5 kg*, as shown in figure, is suspended from two massless springs. When a stone of mass *0.03 kg* is placed on the top of the block, the springs compress an additional *4 cm*. With the stone keeps in touch the block, the springs oscillate with an amplitude of *10 cm.*



(a) What is the frequency of such oscillation?
(b) What is the net force on the stone when it is at the point of maximum upward displacement?
(c) Determine the maximum amplitude of oscillation at which the stone still remains in contact with the block.

Solution

(a) Apply $\sum F_y = 0$ to the stone when it is at its equilibrium position:

$$k\Delta y - mg = 0$$

Solve for k: $k = \dfrac{mg}{\Delta y} = \dfrac{(0.03\,\text{kg})(9.8\,\text{m/s}^2)}{0.04\,\text{m}} = 7.35\,\text{N/m}$

Thus we have $f = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{m+M}} = \dfrac{1}{2\pi}\sqrt{\dfrac{7.35\,\text{N/m}}{0.53\,\text{kg}}} = 0.59\,Hz$

(b) When the stone is at a point of maximum upward displacement: $F_{\text{net}} = mg = (0.03\,\text{kg})(9.8\,\text{m/s}^2) = 0.29N$

(c) The maximum acceleration in terms of the angular frequency and amplitude of the motion: $a_{\max} = A\omega^2$, where $\omega^2 = \dfrac{k}{m+M}$

Thus we have: $a_{\max} = A\dfrac{k}{m+M}$ ....(1)

Set a_{max} = g, equation (1) becomes $g = A(\dfrac{k}{m+M}) = A(\dfrac{7.35}{0.53}) = 13.87(A)$

$$\therefore A = \frac{g}{13.87} = \frac{9.8\,m/s^2}{13.87} = 0.7\ m$$

Example 6

Suppose a spring has the following weird restoring force properties:-

$$F = \begin{cases} -kx & \text{for x } > \text{ 0, stretched} \\ -3kx & \text{for x } < \text{ 0, compressed} \end{cases}$$

An object of mass m is attached to such a weird spring with different spring constants for $x > 0$ and $x < 0$, and is displaced to $x = A$ from its equilibrium position and released.

(a) Determine the period of the oscillation.

(b) Does the period of such oscillation depend on its amplitude A ?

(c) Are the oscillations simple harmonic ?

(d) What is the most negative value of x that the object can reach?

(e) Is the oscillation symmetric about the equilibrium position $x = 0$?

<u>Solution</u>

(a) As the object releases, it approaches the origin, the motion is that a mass being attached to a spring of restoring force constant *k,* and the time to reach the origin is given by $t_1 = \frac{\pi}{2}\sqrt{\frac{m}{k}}$. After passing the equilibrium position (i.e. the origin), the motion is that a mass being attached to a spring of restoring force constant *3k*, and the time it takes to reach the other extreme is given by $t_2 = \frac{\pi}{2}\sqrt{\frac{m}{3k}}$. The period of oscillation is therefore twice the sum of these times, or $T = \pi\sqrt{\frac{m}{k}}(1 + \frac{1}{\sqrt{3}})$.

(b) From the result of part (a), we can conclude that the period of the motion does not depend on the amplitude.

(c) The motion is not simple harmonic.

(d) Denote *A'* to be the most negative extreme. By using the conservation of energy, we have

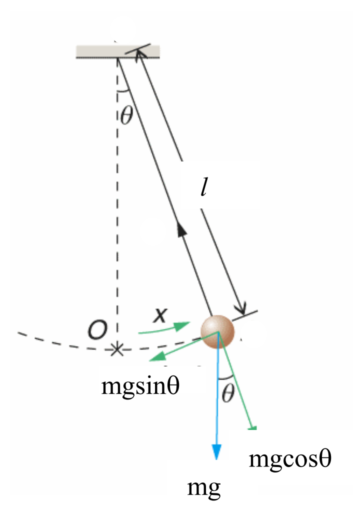
$$\frac{1}{2}kA^2 = \frac{1}{2}(3k)A'^2 \quad \Rightarrow \quad A' = -\frac{A}{\sqrt{3}}$$

(e) The motion is not symmetric about the equilibrium position $x = 0$.

> **Check Point 3:** By using the principle of simple harmonic motion, suggest a method to measure the mass of an object in a 'weightless' condition.

7.9 The simple pendulum

Consider a simple pendulum, which consists of a bob of mass *m* suspended from one end of an unstretchable, massless string of length *L* that is fixed at the other end, as shown in the figure at right. The bob is free to swing back and forth in the plane of the page.



The tangent component of the resultant force $F = -mgsin\theta$

$\square F = ma \Rightarrow -mgsin(\omega t) = ma$

If θ is small, $sin\theta \approx \theta$.

So we have $F = -mgsin\theta \approx -mg\theta = -mgx/l$

$$ma = -mg\frac{x}{l} \Rightarrow m\frac{d^2x}{dt^2} = -\frac{mg}{l}x$$

$$\Rightarrow \frac{d^2x}{dt^2} = -\frac{g}{l}x \quad \Rightarrow \omega^2 = \frac{g}{l}$$

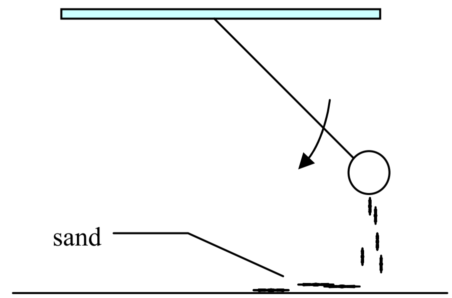
$$\therefore T = 2\pi\sqrt{\frac{l}{g}}$$

Sinθ ~ θ in radian for small θ

θ (degree)	**θ(radian)**	**Sin θ**	**Error**
2	0.03490	0.03490	0.001(%)
3	0.05236	0.05233	0.057(%)
4	0.06981	0.06975	0.086(%)
5	0.08727	0.08716	0.126(%)
10	0.1745	0.1736	0.515(%)
15	0.2618	0.2588	1.14(%)

Note: The period of the simple pendulum is independent of its mass and its amplitude, but depends upon (1) its length and (2) the acceleration due to gravity. On the other hand, notice the approximation $\sin\theta \approx \theta$ made in the above derivation. If the angle is not small, i. e., the initial displacement from point-O is not small compared to the length of the string L, then the motion is periodic but not harmonic, because the restoring force is no longer proportional to θ, but to $\sin\theta$.

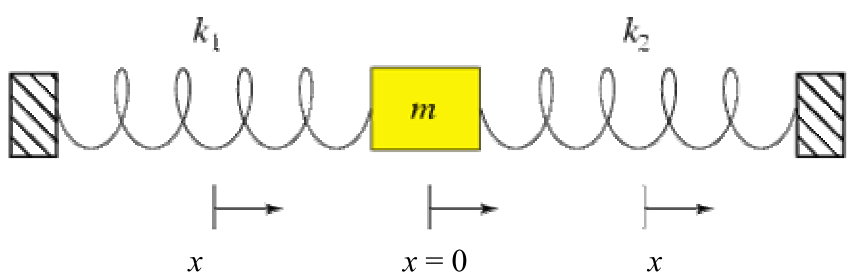
Check Point 4: The pendulum bob as shown in figure leaks sands onto the paper. Is there any effect of this loss of sand on the period of pendulum? Explain.



Check Point 5: A pendulum of length L is suspended from the ceiling of an elevator. The period of pendulum is T when the elevator is at rest. What happens to the period of pendulum when (a) moves upward with constant speed? (b) moves downward with constant acceleration?

7.10 The period of a mass on springs

(1) If a mass is held by two stretched springs, each attached to a wall as shown, show that the motion is *SHM* when M is slightly displaced from its equilibrium position. Also determine its period of oscillation.



Solution

Forces acting on M by two springs are

$F_1 = -k_1x$ due to the spring with force constant k_1; $F_2 = -k_2x$ due to the spring with force constant k_2

Thus the resultant force, $F = F_1 + F_2 = -(k_1 + k_2)x$

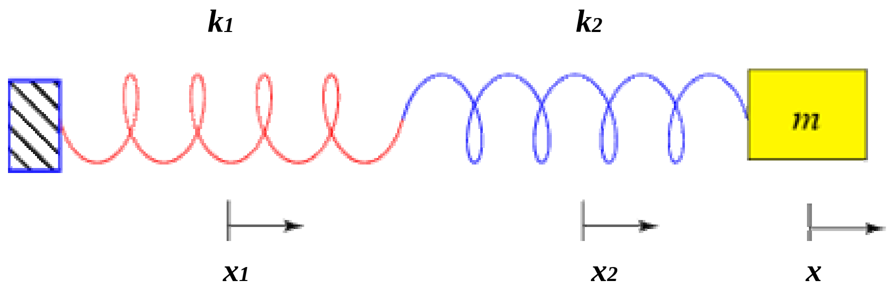
$$\therefore \quad F = Ma = -(k_1 + k_2)x$$

We have $a = -\dfrac{k_1 + k_2}{M}x$

$\therefore$ motion is *SHM*.

The period of motion is $T = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{M}{k_1 + k_2}}$

(2) Two springs placed in series with a mass on the bottom of the second:-
Two springs of force constants k_1 and k_2 are connected to a mass M as shown. Show that the motion of the M is *SHM* and find its period.



Solution

Suppose M is displaced by x to left, then each spring is stretched respectively x_1, x_2 such that the total extension of the spring is given by $x = x_1 + x_2$.
Let the tension be F (same for both)

$F = - k_1 x = - k_2 x$ (tension is equally transmitted throughout the spring)

$$x_1 = \frac{F}{k_1}; \; x_2 = \frac{F}{k_2}$$

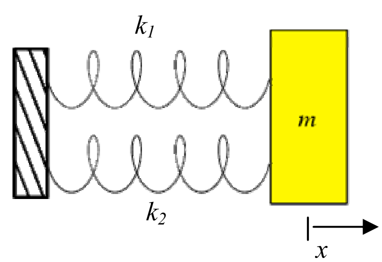
Thus $F = -(K_{eff})x$; where K_{eff} is the effective spring constant of the system.

We have $K = \dfrac{F}{x} = \dfrac{F}{x_1 + x_2} = \dfrac{F}{\dfrac{F}{k_1} + \dfrac{F}{k_2}} = \dfrac{k_1 k_2}{k_1 + k_2}$;   **Note:** $\dfrac{1}{k_{eff}} = \dfrac{1}{k_1} + \dfrac{1}{k_2} + \ldots.$

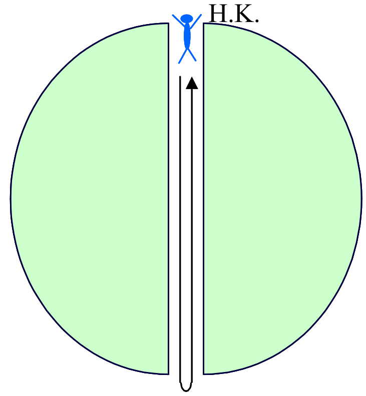
$$\therefore a = -(\frac{k_1 k_2}{k_1 + k_2} \frac{1}{M})x \Rightarrow \text{motion is SHM.}$$

$$\therefore \omega = \sqrt{\frac{k_1 k_2}{(k_1 + k_2)M}} \quad or \quad T = 2\pi \sqrt{\frac{(k_1 + k_2)M}{k_1 k_2}}$$

Check point 6: Show that the motion of object at right is a *SHM*.

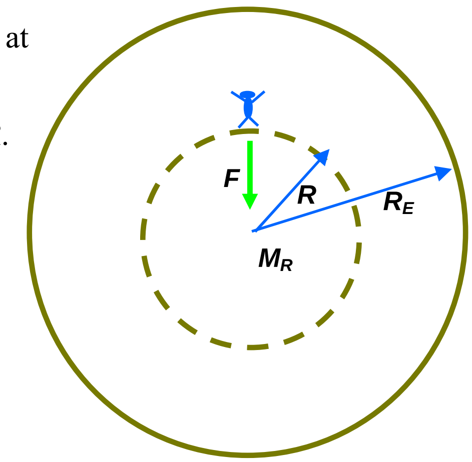


Example 7



An imaginary Transport Tunnel
A straight tunnel is dug from Hong Kong through the center of the Earth and out the other side. A physics student jumps into the hole at noon. (a) Show that the travel of the student in and out of the Earth is simple harmonic motion. (b) What time does he get back to Hong Kong?
Assuming the earth to be a uniform sphere and ignoring the air friction. Given $g = 9.8 m/s^2$, $R_E = 6.38\times10^6\ m$.

Solution



(a) The magnitude of the gravitational force on the student at distance R measured from the center of the earth is

$F_R = G\dfrac{mM_R}{R^2}$, where M_R is the mass inside the radius R.

$$\frac{F_R}{F_{R_E}} = \frac{GmM_R/R^2}{GmM_E/R_E^2} = \frac{M_R}{R^2}\cdot\frac{R_E^2}{M_E} \text{.......}(1)$$

$$\because M \,\alpha\, R^3 \ [M = \rho V = \rho(\tfrac{4}{3}\pi R^3)]$$

thus equation (1) becomes

$$\frac{F_R}{F_{R_E}} = \frac{R^3}{R^2}\cdot\frac{R_E^2}{R_E^3} = \frac{R}{R_E} \text{.......... ....}(2)$$

Since $F_{R_E} = -mg$ ....(3)

Thus we substitute equation (3) into equation (2), which yields

$$F_R = F_{R_E}\cdot\frac{R}{R_E} = -mg\cdot\frac{R}{R_E} = -(k)R \quad \text{where } k = mg/R_E$$

Like a mass on a spring, F is proportional to the displacement but in opposite direction i.e. $F \propto -R$, therefore, the motion of the student in and out of the Earth along the tunnel is *SHM*.

(b) Newton's second law yields

$$\text{F} = m\frac{d^2x}{dt^2} = -\frac{mg}{R_E}\cdot R \quad or \text{ when m is canceled,} \frac{\text{d}^2x}{dt^2} = -(\frac{g}{R_E})\cdot R$$

By comparison with the analogous equation for a spring-mass system, $d^2x/dt^2 = -\omega^2 x = -(g/R_E)x$

$$\therefore \omega = \sqrt{\frac{g}{R_E}}$$

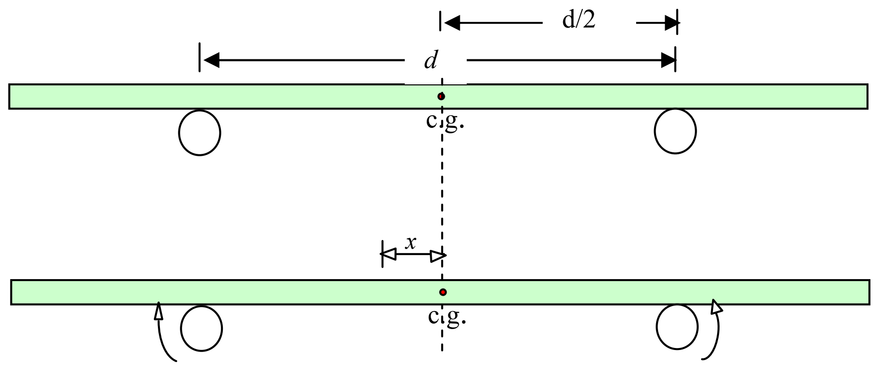
Plug in $g = 9.8 m/s^2$, $R_E = 6.38 \times 10^6\ m$, $\quad T = \dfrac{2\pi}{\omega} = 5067\,s = 84.4\,\text{min}$

Therefore the student gets back to H.K. *84.4 minute*s later at about *1:24 p.m.*

Note: This is the same period as that of a satellite orbiting near the surface of the earth and is independent of the length of the tunnel.

Example 8

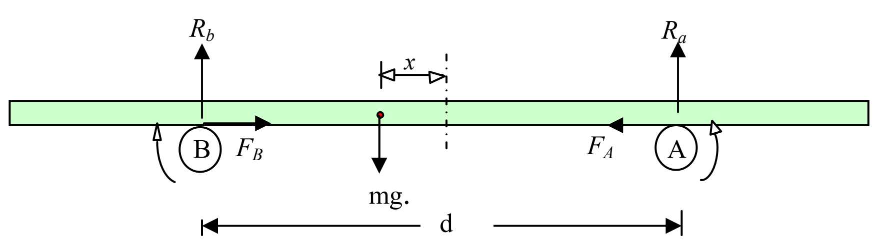
Two discs as shown below, with a distance *d* between the bar's center of mass, can rotate freely about their axes but in opposite directions. A light bar of mass *m* is displaced a small distance *x* from its original position and released. Owing to the unbalanced friction of the two discs, the bar oscillates to and fro on the surface of the discs. Given that the coefficient of kinetic friction between the beam and the discs is μ_k.



(a) Deduce an expression for the reactions on the bar by the discs in terms of *d, x, m* and *g* where *g* is the acceleration due to gravity.
(b) Show that the motion of the bar on the discs is simple harmonic motion.
(c) Determine the period of the resulting motion of the bar.

Solution

(a).Consider the *c.g.* of the bar moved a small distance *x* to the left. Denote R_a, R_b to be the normal reaction acting on the bar due to disc *A* and *B* respectively.



Take moment about point *B*,

$$R_a d = mg(\frac{d}{2} - x) \quad \Rightarrow \quad R_a = mg(\frac{1}{2} - \frac{x}{d})$$

The bar experiences no net force along its vertical direction, so

$$R_b = mg - R_a = mg - mg(\frac{1}{2} - \frac{x}{d}) = mg(\frac{1}{2} + \frac{x}{d})$$

(b) By using Newton's 2nd Law of motion, we have

$F_A - F_B = ma$ where F_A and F_B are the frictional force due to disc A and B.

$\mu_k(R_a - R_b) = ma$

From the result of part (a), we have

$$\mu_k\left[mg(\frac{1}{2} - \frac{x}{d}) - mg(\frac{1}{2} + \frac{x}{d})\right] = ma$$

$$\mu_k(-\frac{2mg}{d})x = ma$$

$$\therefore a = -(\frac{2\mu_k g}{d})x$$

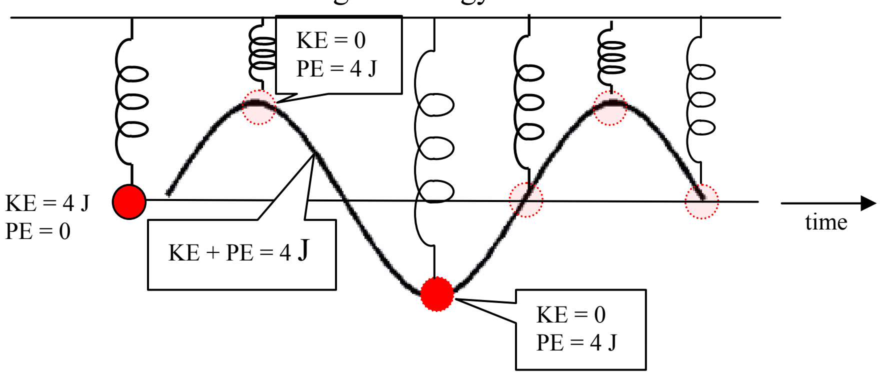
Since the acceleration of the bar a is directly proportional to the its displacement x but in opposite direction, thus the motion of the bar is a simple harmonic motion.

(c) $\omega^2 = \frac{2\mu_k g}{d} \Rightarrow T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{d}{2\mu_k g}}$

> **Check Point 7:** A wooden cube with length L and mass m floats in water with one of its faces parallel to the water surface. If the cube is displaced through a distance y from its equilibrium position, show that the motion of the cube in water is simple harmonic motion and find its period of oscillation. Assume the density of water is ρ.

SHM and energy consideration:

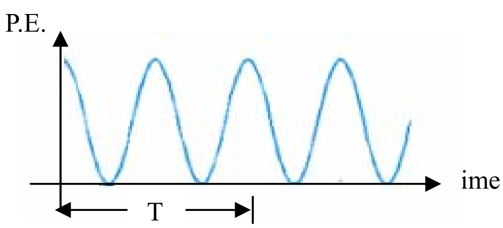
In *SHM*, there is a constant interchange of energy between *KE* and *PE*. The total energy stays constant.

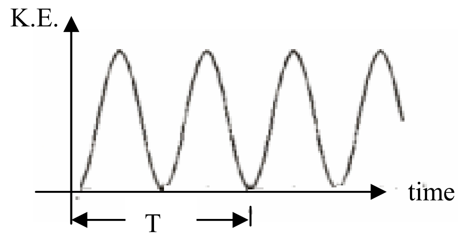


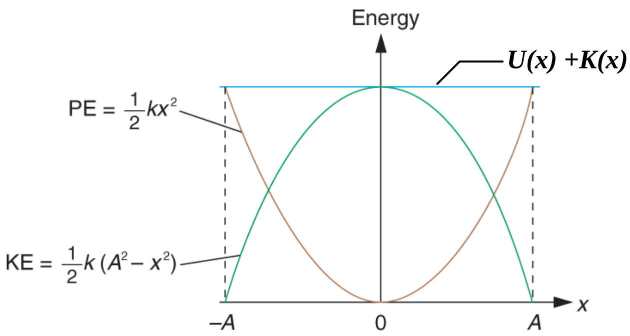
If $x = A\cos\omega t$, then the *KE* is given by

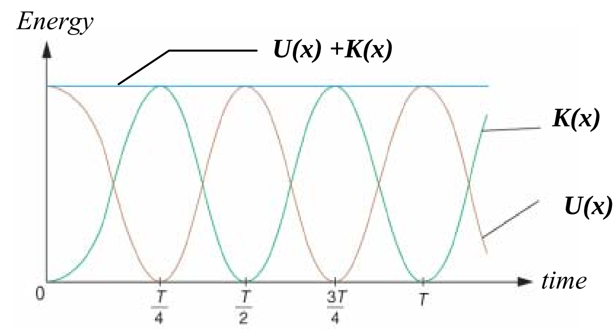
1. Total Energy : $E = 1/2\, kA^2$ is constant. It does not vary with time.
2. Potential Energy : $U = 1/2kx^2 = 1/2kA^2\cos^2\omega t$ where $x = A\cos\,\omega t$.
3. Kinetic Energy : $K = 1/2mv^2 = 1/2m(\omega A\sin\omega t)^2 = 1/2m\omega^2A^2\sin^2\omega t = 1/2kA^2\sin^2\omega t$.

The time variation of energy is therefore as shown below:









Check Point 8: A mass-spring system undergoes *SHM* with an amplitude *A*. Does the total energy change if the mass is doubled but the amplitude is not changed? Explain.

Example 9

Consider a simple harmonic oscillator with $m = 0.5\ kg$, $k = 10\ N/m$ and amplitude $A = 3\ cm$. (a) What is the total energy of the oscillator? (b) What is its maximum speed? (c) What is the speed when $x = 2\ cm$?

Solution

(a). $E = \frac{1}{2}kA^2 = \frac{1}{2}(10)(0.03)^2 = 0.0045J$

(b) $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{10}{0.5}} = 4.47 rad/s \quad \Rightarrow v_{max} = \omega A_{max} = (4.47\, rads^{-1})(0.03\, m) = 0.134\, m/s$

(c) $v = \omega\sqrt{A^2 - x^2} = (4.47\, rads^{-1})\sqrt{(0.03)^2 - (0.02)^2} = 0.1\, m/s$

Example 10

Suppose a particle is attached to an ideal spring. It is found that the particle has a velocity v_1 when its displacement is x_1 and a velocity of v_2 when the displacement is x_2. Find (a) the angular frequency ; (b) the amplitude of the such motion in terms of the given quantities.

Solution

(a) $\frac{1}{2}mv_1^2 + \frac{1}{2}kx_1^2 = \frac{1}{2}m_2^2 + \frac{1}{2}kx_2^2$ where m and k is the mass of the particle and the force constant of the spring respectively.

$$k(x_1^2 - x_2^2) = m(v_2^2 - v_1^2)$$

$$\because \omega = \sqrt{\frac{k}{m}} \qquad \therefore \omega = \sqrt{\frac{v_2^2 - v_1^2}{x_1^2 - x_2^2}}$$

(b) $\frac{1}{2}kA^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}kx_1^2$ where A is the maximum amplitude of the oscillation.

$$A^2 = \frac{m}{k}(v_1^2) + x_1^2 = \frac{x_1^2 - x_2^2}{v_2^2 - v_1^2} \cdot v_1^2 + x_1^2 = \frac{x_1^2 v_2^2 - x_2^2 v_1^2}{v_2^2 - v_1^2}$$

$$\therefore A = \left(\frac{x_1^2 v_2^2 - x_2^2 v_1^2}{v_2^2 - v_1^2} \right)^{1/2}$$

7.11 Resonance, Forced Vibration and Damping

Oscillatory motions are quite common. 3 types of oscillations can be identified, namely: free oscillation, damped oscillation and forced oscillation.

(a) Free oscillation

Definition: total energy is conserved. Amplitude of oscillation is constant. Given example is *SHM*.

(b) Damped oscillation:

In practice, all mechanical oscillations will not perform free oscillations. This is due to presence of dissipative agents that continuously take energy from the system. The oscillation is damped.

Damping may be due to air friction, liquid or solid friction.

Assume damping force is proportional to velocity i.e. $F_d = -bv$

Consider sum of forces: $\Sigma F = -kx - bv$ and $\Sigma F = ma$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \quad \Rightarrow x(t) = A_o e^{-bt/2m}\cos(\omega' t + \delta) \quad \text{where} \quad \omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

1. Slightly damped:

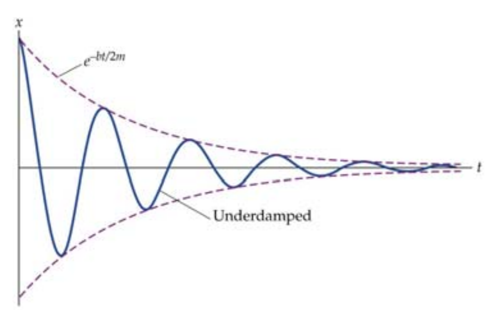
a. The system performs several oscillations and before it comes to rest. The amplitude of oscillation decays *exponentially* with time. i. e., the ratio of amplitude in successive oscillations is constant.

$$\frac{A_2}{A_1} = \frac{A_3}{A_2} = \frac{A_4}{A_3} = .. = K$$

$$\Rightarrow \ln\frac{A_3}{A_1} = \ln(\frac{A_3}{A_2} \times \frac{A_2}{A_1}) = \ln(\frac{A_3}{A_2}) + \ln(\frac{A_2}{A_1}) = 2k$$

$$Similarly, \ln(\frac{A_7}{A_1}) = 7k \quad \therefore \ln(\frac{A_t}{A_1}) = kt$$

Thus $A_t = A_1 e^{-kt}$

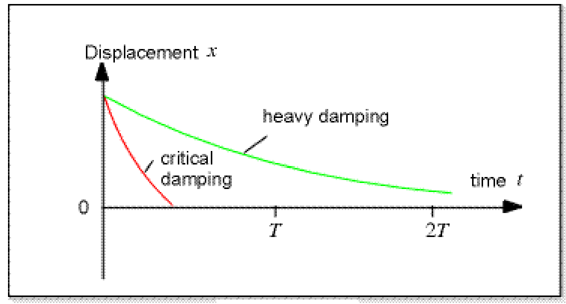


b. The damping force is small amplitude of oscillation gradually decreases, no longer remains constant

c. Period of oscillation however does not change

2. Critically damped

The system does not oscillate and comes to rest in the shortest possible time. A well-designed galvanometer is critically damped – its pointer moves to the new position and then stops without overshooting.



3. Over damping/Heavy damping

a. Damping force is even greater than in critical damping.
b. System does not oscillate but takes longer time than critical damping to return to equilibrium position.

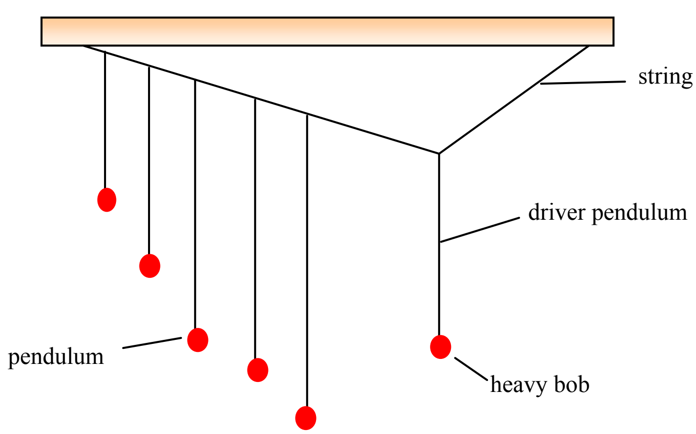
(c) Examples of damped oscillations:

1. Many mechanically vibrating systems do undergo damped oscillation due to presence of internal resistive force and or external resistive forces.
   (a). Due to internal forces in between coils, a stretched spring will lose energy as it vibrates.
   (b). A swinging pendulum gradually slows down due to air friction.
   (c). Vibrating tuning forks or musical instruments – oscillation dies down due to air resistance.
   These are examples of slightly damped motion
2. The paper cone of a loud speaker is made to undergo damped oscillation.
3. Swinging of a metal plate in a magnetic field.
4. Electrical meters such as a moving-coil galvanometer are designed to be critically damped by generation of eddy current so the pointer moves quickly to final steady position.
5. The shock absorber of a motor vehicle employs critical damping to prevent excessive oscillation.

Forced vibration/oscillation

When a vibrating system is subjected to a periodic driving force and set into continuous oscillation, the system is said to undergo forced vibration. Let's look at the following 2 examples.

Example 1: Barton pendulum

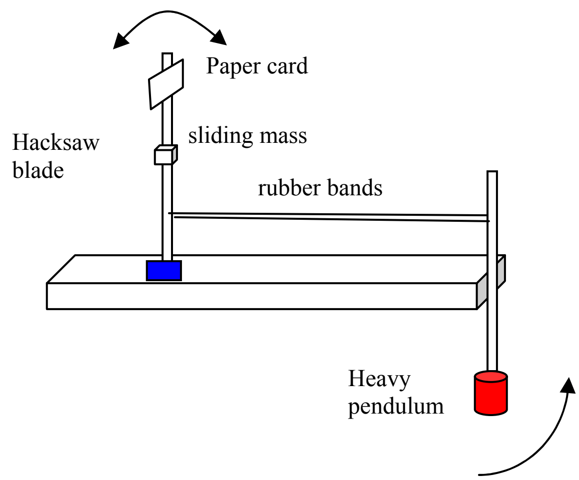


One can make the following observations about forced vibration:

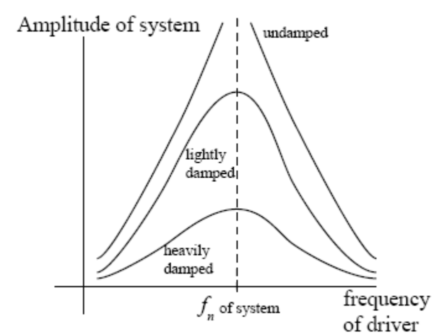
a. Energy is continually transferred from the driver to the driven system. Amplitude of oscillation gradually builds up.
b. Under steady condition, amplitude of vibration is fixed for a fixed driving frequency.
c. This implies the driving force does work on system at the same rate as the system loses energy by doing work against dissipative forces.
d. Amplitude: If the driving frequency is close to natural frequency, the energy transferred is most effective, larger amplitude will build up correspondingly.

Example 2

Hacksaw blade:
Figure as shown at right, a hacksaw blade is forced to oscillate by a heavy pendulum. Under steady conditions, the system oscillates with a frequency equal to that of the external force. The amplitude of vibration is constant. Thus the rate of energy supplied to the system is equal to the rate of energy lost by doing work against dissipative force.



Resonance:
It occurs when the *driving frequency equals the natural frequency of the system*. It is characterized by large oscillations of the system, and thus a *large absorption of energy*.



7.12 Solution to Check Point

1. Simple harmonic motion is the oscillatory motion that occurs when a restoring force in the form of $F = -kx$, acts on an object. The force changes continually as the displacement x changes. A base ball is dropped onto the ground floor. Over and over again, it rebounds to its original height. During the time when the ball is in the air, either falling down or rebounding up, the only force acting on the ball is its weight, which is a constant value. Thus, the motion of the bouncing ball is not simple harmonic motion.

2. The motion eventually stops because energy is lost to overcome air friction.

3. When we want to know a person's weight or mass, we simply get out a scale and stand on it. This works properly on Earth, where gravity is relatively constant and all around us. However, in the orbit, it is more challenging to determine an astronaut's weight or mass. In the orbit, an astronaut is said to be weightless, which is nearly essentially true. With gravity being nearly zero, one has no measurable weight in orbit. However, one still has mass. We cannot simply stand on a scale, since it requires gravity to work. To find an astronaut's mass, without gravity, we can use the Body Mass Measurement Device (BMMD). The

BMMD, as shown below, is a chair that the astronaut sits in. The chair vibrates or oscillates back and forth. Knowing the period of the oscillation, we can determine the mass of someone without the use of gravity. The mass follows from the formula for the period of an oscillating block-spring system.

Astronaut Tamara Jernigan measures her inertial mass aboard the Space Shuttle. (*NASA*)

4. The period of a pendulum is independent of the mass of its bob. Therefore, the period should be unaffected.

5. (a) A pendulum behaves the same in an elevator moving with constant speed as it does in an elevator at rest. Therefore, the period is still T.

   (b) When the elevator moves downward with constant acceleration, the pendulum experiences an effective acceleration of gravity smaller than g. Hence the period increases.

6. The force exerted by two springs attached in parallel to a wall is given by

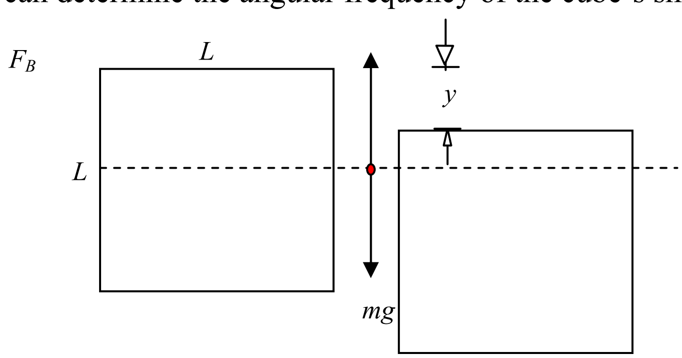
   $F = -(k_1x + k_2x) = -(k_{eff})x.$ The effective spring constant of the system is therefore $k_{eff} = k_1 + k_2$

   By using Newton's 2nd law of motion, we have

   $\sum F = ma = -(k_{\text{eff}})x = -(k_1 + k_2)x$

   Since the acceleration of the system is directly proportional to its displacement but in opposite direction, thus the motion of the block is a simple harmonic motion.

7. The figure shows the forces acting on the cube when it is in equilibrium floating in the water and when it has been pushed down a small distance y. We can find the period of its oscillatory motion from its angular frequency. By applying Newton's 2nd law to the cube, we can obtain its equation of motion; from this equation we can determine the angular frequency of the cube's small-amplitude oscillations.



Apply $\sum T_y = ma_y$ to the cube when it is pushed down a small distance y: $mg - F_B = ma_y$, where F_B is the upthrust acting on the cube.

For $y << 1$, $\Delta F_B \approx dF_B = -\rho V g = -\rho(L^2 y)g = m\dfrac{d^2 y}{dt^2}$

$$\frac{d^2 y}{dt^2} = -\frac{L^2 \rho g}{m} y = -\omega^2 y \qquad \text{where} \quad \omega^2 = \frac{L^2 \rho g}{m}$$

Thus the period of oscillation is given by $T = \dfrac{2\pi}{L\sqrt{\dfrac{\rho g}{m}}} = \dfrac{2\pi}{L} \cdot \sqrt{\dfrac{m}{\rho g}}$

8. The total energy of this system increases by factor of two. This follows because doubling the mass causes the kinetic energy to be doubled at all times. But the total energy of the system is equal to the maximum kinetic energy; therefore, the total energy is also doubled.