[Solution]  

Theoretical Question 2  

Motion of an Electric Dipole in a Magnetic Field

(1) Conservation Laws

\[ (1a) \quad \vec{r}_{CM} = \frac{1}{2} (\vec{r}_1 + \vec{r}_2), \quad \vec{v}_{CM} = \frac{1}{2} (\vec{v}_1 + \vec{v}_2), \quad \vec{l} = \vec{r}_1 - \vec{r}_2, \quad \vec{u} = \dot{\vec{l}} = \vec{v}_1 - \vec{v}_2 \]

Total force \( \vec{F} \) on the dipole is

\[ \vec{F} = \vec{F}_1 + \vec{F}_2 = q(\vec{E} + \vec{v}_1 \times \vec{B}) + (-q)(\vec{E} + \vec{v}_2 \times \vec{B}) = q(\vec{v}_1 - \vec{v}_2) \times \vec{B} \]

\[ = q \vec{\ell} \times \vec{B} \]

so that

\[ M \dot{\vec{v}}_{CM} = q \vec{\ell} \times \vec{B} \quad (M = 2m) \quad (1) \]

Computing the torque for rotation around the center of mass, we obtain

\[ I \dot{\omega} = \frac{1}{2} (\vec{i} \times (q\vec{v}_1 \times \vec{B})) + \frac{1}{2} (\vec{-i} \times (-q\vec{v}_2 \times \vec{B})) \]

\[ = q \vec{\ell} \times (\vec{v}_{CM} \times \vec{B}) \quad (2) \]

where

\[ I = \frac{1}{2} m\ell^2 \quad (3) \]

(1b) From eq.(1), we obtain the conservation law for the momentum:

\[ \dot{\vec{P}} = 0, \quad \vec{P} = M \vec{v}_{CM} - q \vec{\ell} \times \vec{B} \quad (4) \]

From eq.(1) and eq.(2), one obtains the conservation law for the energy.

\[ \dot{E} = 0, \quad E = \frac{1}{2} M \dot{\vec{v}}_{CM}^2 + \frac{1}{2} I \omega^2 \quad (5) \]

(1c) Using eq.(4) and eq.(2),

\[ \frac{d}{dt} (\vec{r}_{CM} \times \vec{P}) \cdot \hat{B} = (\vec{v}_{CM} \times \vec{P}) \cdot \hat{B} = -q \vec{v}_{CM} \times (\vec{\ell} \times \vec{B}) \cdot \hat{B} \]

\[ = q(\vec{\ell} \times \vec{B}) \times \vec{v}_{CM} \cdot \hat{B} = q(\vec{\ell} \times \vec{B}) \cdot (\vec{v}_{CM} \times \hat{B}) \]

\[ = q \vec{l} \cdot (\vec{B} \times (\vec{v}_{CM} \times \hat{B})) = -q \vec{l} \cdot ((\vec{v}_{CM} \times \hat{B}) \times \vec{B}) \]

\[ = -q \vec{l} \times (\vec{v}_{CM} \times \vec{B}) \cdot \hat{B} \]

\[ = -I \dot{\omega} \cdot \hat{B} \]
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we obtain the conservation law

\[ \dot{\mathbf{J}} = 0 \quad \mathbf{J} = (\mathbf{r}_{CM} \times \mathbf{P} + I \mathbf{\omega}) \cdot \mathbf{B} \]  \hspace{1cm} (6)

for the component of the angular momentum along the direction of \( \mathbf{B} \).

(2) Motion in a Plane Perpendicular to \( \mathbf{B} \)

(2a) Write

\[ \ell = \ell \{ \cos \varphi (t) \mathbf{x} + \sin \varphi (t) \mathbf{y} \}, \quad \varphi (0) = 0, \quad \varphi (0) = \omega_0 \]  \hspace{1cm} (7)

Note that

\[ \mathbf{\omega} = \dot{\varphi} \mathbf{\hat{z}} \]  \hspace{1cm} (8)

From eq.(4), we have

\[ M \mathbf{\ddot{v}}_{CM} = \mathbf{\dot{P}} + q \ell B (\sin \varphi \mathbf{\hat{x}} - \cos \varphi \mathbf{\hat{y}}) \]  \hspace{1cm} (9)

At \( t = 0 \), we have \( v_{CM} = 0, \quad \varphi = 0 \) so that

\[ \mathbf{\dot{P}} = q \ell B \mathbf{\hat{y}} \]  \hspace{1cm} (10)

Hence from eqs.(9) and (10) we have

\[ \mathbf{x}_{CM} = \left( \frac{q \ell B}{M} \right) \sin \varphi, \quad \mathbf{y}_{CM} = \left( \frac{q \ell B}{M} \right) (1 - \cos \varphi) \]  \hspace{1cm} (11)

From conservation of energy, i.e. Eq.(5), we have

\[ \frac{1}{2} I \dot{\varphi}^2 + \frac{(q \ell B)^2}{M} (1 - \cos \varphi) = \frac{1}{2} I \omega_0^2 \]

\[ \therefore \quad \dot{\varphi}^2 + \frac{1}{2} \omega_c^2 (1 - \cos \varphi) = \omega_0^2 \]  \hspace{1cm} (12)

where

\[ \omega_c^2 = \frac{4(q \ell B)^2}{MI} = \left( \frac{2qB}{m} \right)^2 \]  \hspace{1cm} (13)

In order to make a full turn, \( \varphi \) can not become zero so that
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\[ \omega_0^2 > \omega_c^2 \Rightarrow |\omega_0| > \omega_c = \frac{2qB}{m} \]

\[ (14) \]

(2b) From Eq.(6), we have

\[ x_{CM}P + I\omega = J \]

where \( P \) is the magnitude of \( \vec{P} \).

At \( t = 0 \), we have \( J = I\omega_0 \) so that

\[ x_{CM}P + I\omega = I\omega_0 \]

\[ (16) \]

From eq.(12), one can see that \( \omega_0^2 \geq \omega^2 \) so that \( x_{CM} \geq 0 \). Thus \( x_{CM} \) reaches a maximum \( d_m \) when \( \omega \) takes its minimum value.

When \( \omega_0 < \omega_c \), the minimum value of \( \omega \) is \( -\omega_0 \) so that

\[ d_m = \frac{2I}{P} \omega_0 = \left( \frac{m\omega_0}{qB} \right) \ell, \quad \omega_0 < \omega_c \]

\[ (17) \]

When \( \omega_0 > \omega_c \), the minimum value of \( \omega \) is \( \sqrt{\omega_0^2 - \omega_c^2} \) so that

\[ d_m = \left( \frac{I}{P} \right) \left( \omega_0 - \sqrt{\omega_0^2 - \omega_c^2} \right) = \frac{m}{2qB} \left( \omega_0 - \sqrt{\omega_0^2 - \omega_c^2} \right) \ell, \quad \omega_0 > \omega_c \]

\[ (18) \]

When \( \omega_0 = \omega_c \), \( \omega^2 = \frac{1}{2} \omega_c^2 \left( 1 + \cos \phi \right) = \omega_c^2 \cos^2 \frac{\phi}{2} \)

\[ \therefore \phi = \omega_c \cos \frac{\phi}{2} \]

When \( \phi \) is close to \( \pi \), let \( \phi = \pi - 2\epsilon \) then

\[ \dot{\epsilon} = -\frac{1}{2} \omega_c \sin \epsilon \approx -\frac{1}{2} \omega_c \epsilon \]

\[ \therefore \epsilon \sim e^{-\omega_c t/2} \]

so that it will take \( t \rightarrow \infty \) for \( \epsilon \rightarrow 0 \), i.e. for \( \phi \) to reach \( \pi \). Hence

\[ d_m = \left( \frac{I}{P} \right) \omega_c = \left( \frac{m\omega_c}{2qB} \right) \ell, \quad \omega_0 = \omega_c \]

\[ (19) \]

(2c) Tension on the rod comes from three sources:
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(i) Coulomb force = \( \frac{1}{4\pi \varepsilon_0} \frac{q^2}{\ell^2} \)  \( \text{(20)} \)

Positive value means compression on the rod.

(ii) Centrifugal force due to rotation of the rod = \( -\frac{1}{2} m\omega^2 \ell \)  \( \text{(21)} \)

(iii) Magnetic force on the particles due to the motion of the center of the mass

\[
= q\vec{v}_{CM} \times \vec{B} \cdot (-\vec{\ell}) = q\vec{v}_{CM} \cdot \vec{\ell} \times \vec{B}
\]

Taking the square of both sides of eq.(4) and using the initial condition for the value of \( P^2 \), we obtain

\[
\frac{1}{2} M\dot{\ell}^2_{CM} = q\ell \vec{v}_{CM} \cdot \vec{\ell} \times \vec{B} = \frac{1}{2} I (\omega_0^2 - \omega^2)
\]

Combining the three forces, we have

\[
tension \ on \ the \ rod = \frac{1}{4\pi \varepsilon_0} \frac{q^2}{\ell^2} - \frac{1}{2} m\ell \omega^2 + \frac{1}{4} m\ell (\omega_0^2 - \omega^2)
\]

A positive value means compression on the rod.