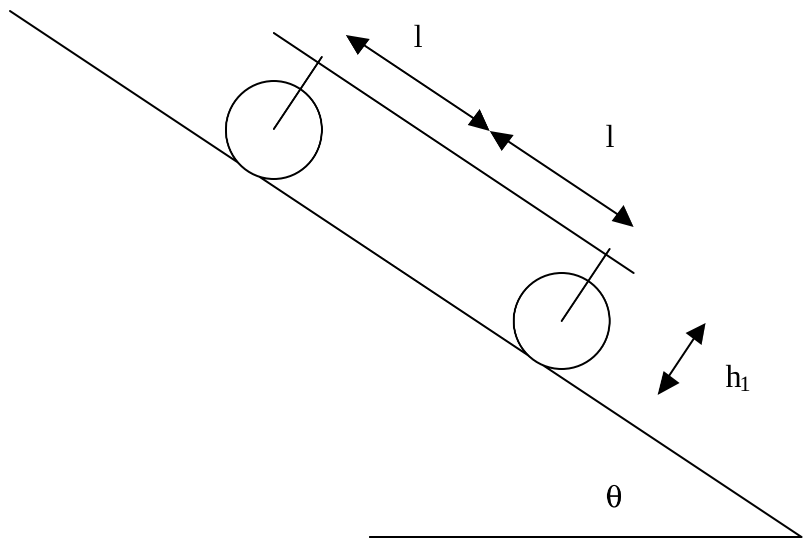
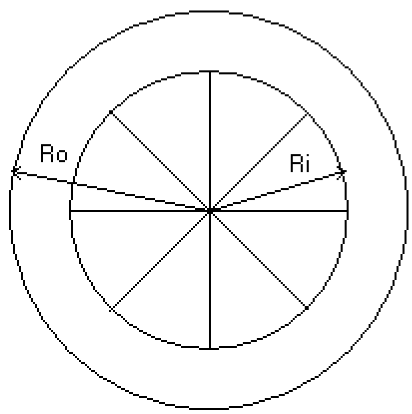
SOLUTION T3 : . **A Heavy Vehicle Moving on An Inclined Road**



To simplify the model we use the above figure with $h_1 = h+0.5\ t$
$R_o = R$

1. **Calculation of the moment inertia of the cylinder**

R_i=0.8 R_o
Mass of cylinder part : $m_{cylinder}$ =0.8 M
Mass of each rod : m_{rod} = 0.025 M



$$I = \oint_{wholepart} r^2 dm = \oint_{cyl.shell} r^2 dm + \oint_{\text{rod1}} r^2 dm + ... + \oint_{rod\,n} r^2 dm \qquad 0.4 \text{ pts}$$

$$\oint_{cyl.shell} r^2 dm = 2\boldsymbol{ps}\int_{Ri}^{Ro} r^3 dr = 0.5\boldsymbol{ps}(R_o^4 - R_i^4) = 0.5m_{cylinder}(R_o^2 + R_i^2)$$

$$= 0.5(0.8M)R^{\ 2}(1+0.64) = 0.656MR^{\ 2} \qquad 0.5 \text{ pts}$$

$$\oint_{\text{rod}} r^2 dm = \boldsymbol{l}\int_0^{Rin} r^2 dr = \frac{1}{3}\,\boldsymbol{l}R_{in}^3 = \frac{1}{3}m_{rod}R_{in}^2 = \frac{1}{3}0.025M(0.64R^2) = 0.00533MR^2 \qquad 0.5 \text{ pts}$$

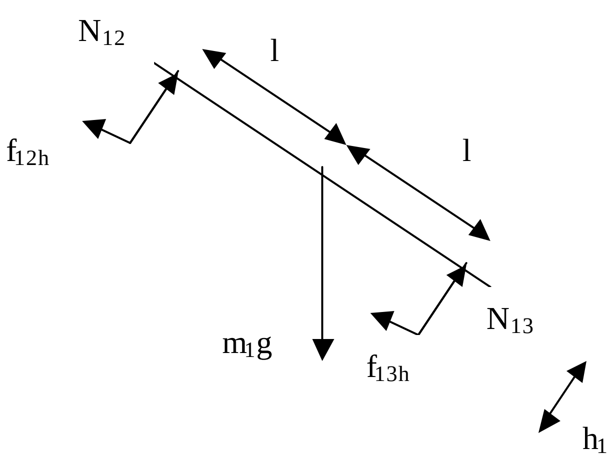
The moment inertia of each wheel becomes

$$I = 0.656MR^2 + 8x0.00533MR^2 = 0.7MR^2 \qquad 0.1 \text{ pts}$$

2. Force diagram and balance equations:

To simplify the analysis we devide the system into three parts: frame (part1) which mainly can be treated as flat homogeneous plate, rear cylinders (two cylinders are treated collectively as part 2 of the system), and front cylinders (two front cylinders are treated collectively as part 3 of the system).

Part 1 : Frame



0.4 pts

The balance equation related to the forces work to this parts are:

Required conditions:
Balance of force in the horizontal axis

$m_1 g \sin q - f_{12h} - f_{13h} = m_1 a$ (1) 0.2 pts

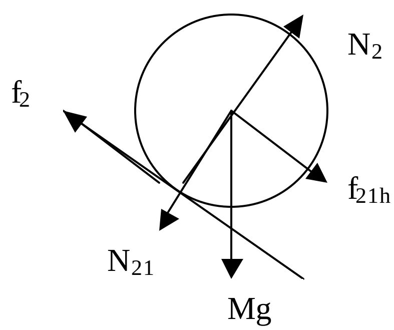
Balance of force in the vertical axis

$m_1 g \cos q = N_{12} + N_{13}$ (2) 0.2 pts

Then torsi on against O is zero, so that

$N_{12} l - N_{13} l + f_{12h} h_1 + f_{13h} h_1 = 0$ (3) 0.2 pts

Part two : Rear cylinder



0.25 pts

From balance condition in rear wheel :

$f_{21h} - f_2 + Mg \sin q = Ma$ (4) 0.15 pts

$N_2 - N_{21} - Mg \cos q = 0$ (5) 0.15 pts

For pure rolling:

$$f_2 R = I a_2 = I \frac{a_2}{R}$$

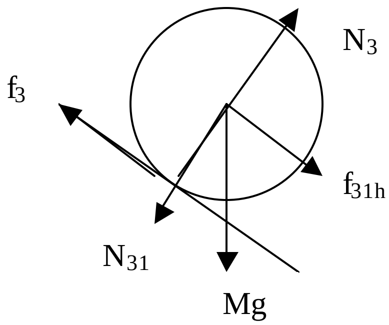
$$\text{or} \quad f_2 = \frac{I}{R^2} a \qquad (6)$$

For rolling with sliding:

$F_2 = u_k N_2$ (7)

0.2 pts

Part Three : Front Cylinder:



0.25 pts

From balance condition in the front whee l :

$f_{31h} - f_3 + Mg\sin\theta = Ma$ (8) 0.15 pts

$N_3 - N_{31} - Mg\cos\theta = 0$ (9) 0.15 pts

For pure rolling:

$$f_3 R = I\alpha_3 = I\frac{a_3}{R}$$

or $f_3 = \dfrac{I}{R^2}a$ (10)

For rolling with sliding:

$F_3 = u_k N_3$ (11)

0.2 pts

3. From equation (2), (5) and (9) we get

$m_1 g\cos\theta = N_2 - m_2 g\cos\theta + N_3 - m_3 g\cos\theta$

$N_2 + N_3 = (m_1 + m_2 + m_3)g\cos\theta = 7Mg\cos\theta$ (12)

And from equation (3), (5) and (8) we get

$(N_3 - Mg\cos\theta)\,l - (N_2 - Mg\cos\theta)\,l = h_1(f_2 + Ma - Mg\sin\theta + f_3 + Ma - Mg\sin\theta)$

$(N_3 - N_2) = h_1(f_2 + 2Ma - 2Mg\sin\theta + f_3)/l$

Equations 12 and 13 are given **0.25 pts**

CASE ALL CYLINDER IN PURE ROLLING

From equation (4) and (6) we get

$f_{21h} = (I/R^2)a + Ma - Mg \sin\theta$ (14) 0.2 pts

From equation (8) and (10) we get

$f_{31h} = (I/R^2)a + Ma - Mg \sin\theta$ (15) 0.2 pts

Then from eq. (1) , (14) and (15) we get

$5Mg \sin\theta - \{(I/R^2)a + Ma - Mg \sin\theta\} - \{(I/R^2)a + Ma - Mg \sin\theta\} = m_1 a$

$7 Mg \sin\theta = (2I/R^2 + 7M)a$

$$a = \frac{7Mg\sin\theta}{7M + 2\frac{I}{R^2}} = \frac{7Mg\sin\theta}{7M + 2\frac{0.7MR^2}{R^2}} = 0.833 g \sin\theta \qquad (16) \quad 0.35 \text{ pts}$$

$$N_3 = \frac{7M}{2} g\cos\theta + \frac{h_1}{l}[(M + \frac{I}{R^2}) \times 0.833 g \sin\theta - Mg\sin\theta]$$
$$= 3.5\text{Mg}\cos\theta + \frac{h_1}{l}[(M + 0.7M) \times 0.833 g \sin\theta - Mg\sin\theta]$$
$$= 3.5 \text{ Mg} \cos\theta + 0.41\frac{h_1}{l} Mg\sin\theta$$

$$N_2 = \frac{7M}{2} g\cos\theta - \frac{h_1}{l}[(\frac{I}{R^2} + M) \times 0.833 g \sin\theta - Mg\sin\theta]$$
$$= 3.5\text{g}\cos\theta - \frac{h_1}{l}[(0.7M + M)\frac{7Mg\sin\theta}{0.7M + 7M} - 2Mg\sin\theta]$$
$$= 3.5\text{g}\cos\theta - 0.41\frac{h_1}{l} Mg\sin\theta$$

0.2 pts

The Conditions for pure rolling:

$f_2 \le \mu_s N_2$ and $f_3 \le \mu_s N_3$

$\frac{I_2}{R_2^2} a \le \mu_s N_2$ and $\frac{I_3}{R_3^2} a \le \mu_s N_3$ 0.2 pts

The left equation becomes

$$0.7M \times 0.833 g \sin\theta \le \mu_s (3.5\text{Mg}\cos\theta - 0.41\frac{h_1}{l} Mg\sin\theta)$$

$$\tan\theta \le \frac{3.5\mu_s}{0.5831 + 0.41\mu_s \frac{h_1}{l}}$$

While the right equation becomes

$$0.7m \times 0.833 g \sin\theta \le \mu_s (3.5mg\cos\theta + 0.41\frac{h_1}{l} mg\sin\theta)$$

$$\tan\theta \le \frac{3.5\mu_s}{0.5831 - 0.41\mu_s \frac{h_1}{l}}$$

(17) 0.1 pts

CASE ALL CYLINDER SLIDING

From eq. (4) $f_{21h} = Ma + u_k N_2 - Mg\sin\theta$ (18) 0.15 pts

From eq. (8) $f_{31h} = Ma + u_k N_3 - Mg\sin\theta$ (19) 0.15 pts

From eq. (18) and 19 :

$5Mg\sin\theta - (Ma + u_k N_2 - Mg\sin\theta) - (Ma + u_k N_3 - Mg\sin\theta) = m_1 a$

$$a = \frac{7Mg\sin\theta - \mu_k N_2 - \mu_k N_3}{7M} = g\sin\theta - \frac{\mu_k (N_2 + N_3)}{7M} \qquad (20) \qquad 0.2 \text{ pts}$$

$$N_3 + N_2 = 7Mg\cos\theta$$

From the above two equations we get :

$$a = g\sin\theta - \mu_k g\cos\theta \qquad 0.25 \text{ pts}$$

The Conditions for complete sliding: are the opposite of that of pure rolling

$$f_2 > \mu_s N'_2 \quad \text{and} \quad f_3 > \mu_s N'_3$$

$$\frac{I_2}{R_2^2} a > \mu_s N'_2 \quad \text{and} \quad \frac{I_3}{R_3^2} a > \mu_s N'_3 \qquad (21) \qquad 0.2 \text{ pts}$$

Where N_2' and N_3' is calculated in case all cylinder in pure rolling. 0.1 pts

Finally weget

$$\tan\theta > \frac{3.5\mu_s}{0.5831 + 0.41\mu_s \frac{h_1}{l}} \quad \text{and} \quad \tan\theta > \frac{3.5\mu_s}{0.5831 - 0.41\mu_s \frac{h_1}{l}} \qquad 0.2 \text{ pts}$$

The left inequality finally become decisive.

CASE ONE CYLINDER IN PURE ROLLING AND ANOTHER IN SLIDING CONDITION

{ For example R_3 (front cylinders) pure rolling while R_2 (Rear cylinders) sliding}

From equation (4) we get

$F_{21h} = m_2 a + u_k N_2 - m_2 g \sin\theta$ (22) 0.15 pts

From equation (5) we get

$f_{31h} = m_3 a + (I/R^2)a - m_3 g \sin\theta$ (23) 0.15 pts

Then from eq. (1) , (22) and (23) we get

$m_1 g \sin\theta - \{ m_2 a + u_k N_2 - m_2 g \sin\theta\} - \{m_3 a + (I/R^2)a - m_3 g \sin\theta\} = m_1 a$

$m_1 g \sin\theta + m_2 g \sin\theta + m_3 \sin\theta - u_k N_2 = (I/R^2 + m_3)a + m_2 a + m_1 a$

$5Mg \sin\theta + Mg \sin\theta + Mg \sin\theta - u_k N_2 = (0.7M + M)a + Ma + 5Ma$

$$a = \frac{7Mg \sin q - m_k N_2}{7.7M} = 0.9091 g \sin q - \frac{m_k N_2}{7.7M} \quad (24) \quad 0.2 \text{ pts}$$

$$N_3 - N_2 = \frac{h_1}{l}\left(m_k N_2 + \frac{I}{R^2} a + 2Ma - 2Mg \sin q\right)$$

$$N_3 - N_2 = \frac{h_1}{l}\left(m_k N_2 + 2.7M \times 0.9091 g \sin q - 2.7 m_k N_2 / 7.7 - 2Mg \sin q\right)$$

$$N_3 - N_2\left(1 + 0.65 m_k \frac{h_1}{l}\right) = 0.4546 Mg \sin q$$

$$N_3 + N_2 = 7Mg \cos\theta$$

Therefore we get

$$N_2 = \frac{7Mg \cos q - 0.4546 Mg \sin q}{2 + 0.65 m_k \frac{h_1}{l}}$$

$$N_3 = 7Mg \cos q - \frac{7Mg \cos q - 0.4546 Mg \sin q}{2 + 0.65 m_k \frac{h_1}{l}} \quad (25) \quad 0.3 \text{ pts}$$

Then we can substitute the results above into equation (16) to get the following result

$$a = 0.9091 g \sin q - \frac{m_k N_2}{7.7M} = 0.9091 g \sin q - \frac{m_k}{7.7} \frac{7g \cos q - 0.4546 g \sin q}{2 + 0.65 m_k \frac{h_1}{l}} \quad (26)$$

0.2 pts

The Conditions for this partial sliding is:

$$f_2 \le m_s N_2' \qquad \text{and} \quad f_3 \rangle m_s N_3'$$

$$\frac{\mathrm{I}}{\mathrm{R}^2}\mathrm{a} \le m_s N_2' \qquad \text{and} \quad \frac{\mathrm{I}}{\mathrm{R}^2}\mathrm{a} \rangle m_s N_3' \qquad (27) \qquad 0.25 \text{ pts}$$

where N_2' and N_3' are normal forces for pure rolling condition

4. Assumed that after rolling d meter all cylinder start to sliding until reaching the end of incline road (total distant is s meter). Assummed that ηmeter is reached in t_1 second.

$$v_{t1} = v_o + at_1 = 0 + a_1 t_1 = a_1 t_1$$

$$d = v_o t_1 + \frac{1}{2} a_1 t_1^2 = \frac{1}{2} a_1 t_1^2$$

$$t_1 = \sqrt{\frac{2d}{a_1}}$$

0.5 pts

$$v_{t1} = a_1 \sqrt{\frac{2d}{a_1}} = \sqrt{2da_1} = \sqrt{2d0.833 g \sin \boldsymbol{q}} = \sqrt{1.666 dg \sin \boldsymbol{q}} \qquad (28)$$

The angular velocity after rolling d meters is same for front and rear cylinders:

$$\boldsymbol{w}_{t1} = \frac{v_{t1}}{R} = \frac{1}{R}\sqrt{1.666 dg \sin \boldsymbol{q}} \qquad (29)$$

0.5 pts

Then the vehicle sliding untill the end of declining road. Assumed that the time needed by vehicle to move from d position to the end of the declining road is t_2 second.

$$v_{t2} = v_{t1} + a_2 t_2 = \sqrt{1.666 dg \sin \boldsymbol{q}} + a_2 t_2$$

$$s - d = v_{t1} t_2 + \frac{1}{2} a_2 t_2^2$$

$$t_2 = \frac{-v_{t1} + \sqrt{v_{t1}^2 + 2a_2(s-d)}}{a_2} \qquad (30) \qquad 0.4 \text{ pts}$$

$$v_{t2} = \sqrt{1.666 dg \sin \boldsymbol{q}} - v_{t1} + \sqrt{v_{t1}^2 + 2a_2(s-d)}$$

Inserting v_{t1} and a_2 from the previous results we get the final results.

For the angular velocity, while sliding they receive torsion:

$$t = m_k NR$$

$$a = \frac{t}{I} = \frac{m_k NR}{I} \qquad (31)$$

$$w_{t2} = w_{t1} + a t_2 = \frac{1}{R}\sqrt{1.666\, dg \sin q} + \frac{m_k NR}{I} \frac{-v_{t1} + \sqrt{v_{t1}^2 + 2a_2 (s-d)}}{a_2}$$

0.6 pts