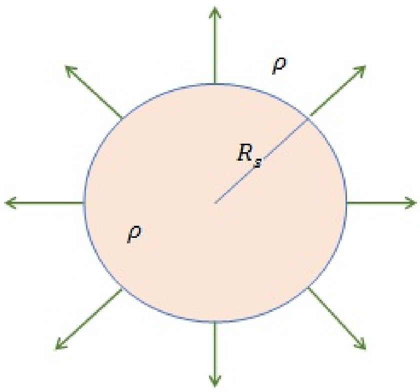
Cosmic Inflation

Due to the relative movement of galaxies observed from the earth, the wavelength of visible spectrum of a particular galaxy differs from its original wavelength, which is known as the electromagnetic Doppler effect. One expects, for a collection of galaxies, to a random distributions of wavelength shifts: some positive (red shift) and some negative (blue shift). However, observations show that all, expect for a nearby group of galaxies, are red shifted. This must be true even if the observation take place on different point in the universe. As a conclusion, our universe must be expanding. On the other hand local irregularity of the universe can be neglected on scales of more than 100 Mpc, in which 1 pc = 3.26 light-years. Averaged over large scales, the clumpy distribution of galaxies becomes more and more isotropic (independent of direction) and homogeneous (independent of position). Therefore we can assume the universe as a matter having a uniform mass density ρ and is expanding.

A. Expansion of Universe



For a simple model of our universe, let us consider an expanding uniform-density sphere embedded in a medium of a much larger sphere with the same density. Let say at some time, the radius of the smaller sphere is R_s. To express the expansion of the sphere, the time dependency of the radius $R(t)$ can be expressed by scale factor $a(t)$, that is $R(t) = a(t)R_s$.

Using Newton's law of gravity to evaluate velocity of a mass element on the sphere boundary according the model of our universe, one can obtain a set of Friedmann equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = A_1\rho(t) - \frac{kc^2}{R_s^2 a^2(t)} \qquad (1)$$

where k a dimensionless constant, and c is velocity of light.

A.1	Determine the constant A_1 in the equation (1)	1.3 pt.

The discussion so far is non-relativistic. But in fact, it can be extended to a relativistic system by reinterpreting $\rho(t)c^2$ as total energy density (excluding the gravitational potential energy). In this relativistic system derives the 2nd Friedmann equation:

$$\dot{\rho} + A_2 \left(\rho + \left(\frac{p}{c^2}\right)\right) \frac{\dot{a}}{a} = 0 \qquad (2)$$

using the 1st law thermodynamics of an adiabatic system, where p denotes the pressure on the sphere.

A.2	Determine the constants A_2 in the equation (2)	0.9 pt.

To solve Eqs. (1) and (2), one should assume a relation $p = p(\rho)$, such as $p(t)/c^2 = w\rho(t)$, where w is a constant. There is also a factor $H = \dot{a}/a$ being called Hubble parameter. The present values of parameters are usually symbolized by subscript 0 such as t_0, ρ_0, H_0, a_0 and so on. For simplicity, we take $a_0 = 1$.

Universe is believed to start from a big explosion called Big-Bang that produces radiation of relativistic particles. During its expansion, the universe is cooling down and the particles in it become non-relativistic. However, the recent observations clarify that the present universe is dominated by cosmological constant energy density. For the case of photon, as the universe is expanding, the photon's wavelength expands proportionally to the scale factor.

A.3	For each of the following three cases determine the resulting value of w: (i) a universe filled only with radiation (i.e. photon energy), (ii) a universe filled only with non-relativistic matter and (iii) a universe with constant energy density.	1.2 pt.
A.4	In the case of $k = 0$, find $a(t)$ for each case of (i) to (iii) being mentioned in A.3. Use the initial condition $a(t = 0) = 0$ for case (i) and (ii), and use the condition $a_0 = 1$ for case (iii).	1.2 pt.

Constant k in Eq. (1) refers to classification of spatial geometry of the universe. Its value can be $k = +1$ for positive-curvature universe (closed), $k = 0$ for flat universe (infinite), and $k = -1$ for negative-curvature universe (open, infinite). Let define a density ratio $\Omega = \rho/\rho_c$, where $\rho_c c^2 = H^2/A_1$ is critical energy density. Note that A_1 is obtained from problem A.1.

A.5	Express k in Eq.(1) in terms of Ω, H, a, and R_0.	0.1 pt.
A.6	Find a range for Ω that corresponds to each value of $k = +1$, $k = 0$ and $k = -1$.	0.3 pt.

B. Motivation To Introduce Inflation Phase and Its General Conditions

The observation of cosmic microwave background radiation (CMB) suggests that our present universe is approximately flat. The problem is that if this is true then the present universe should start from exactly flat early universe, otherwise any deviation from the flatness will eventually grow over time and spoil the present flatness.

B.1	Find $(\Omega(t) - 1)$ as a function of time for the universe when it is dominated by radiation or non-relativistic matter (see problem A.3).	0.5 pt.

To solve the problem, at some early time in its history, the universe should undergo a constant energy density domination period which leads to an exponential expansion so called inflation period.

B.2	For this constant energy density domination period, find $(\Omega(t) - 1)$ as a function of time. Assume that $(\Omega(t) - 1) \ll 1$.	0.3 pt.
B.3	Show that condition for inflation implies several following conditions: negative pressure, accelerated expansion $(\ddot{a} > 0)$, and decreasing Hubble radius $(d(aH)^{-1}/dt < 0)$.	0.9 pt.
B.4	Show that the condition of decreasing Hubble radius can be expressed in terms of parameter $\epsilon = -\dot{H}/H^2$ as $\epsilon < 1$.	0.2 pt.

Inflation occurs as long as $\epsilon < 1$ and then ends when $\epsilon = 1$. We can define e-folding number N, such that $dN = d\ln a = Hdt$ and $N = 0$ at the end of inflation.

C. Inflation Generated by Homogenously Distributed Matter

As an example of simple physical system that can generate period of inflation is a universe dominated by homogenously distributed matter. This kind of matter is called inflaton and can be characterized by a function $\phi(t)$.

The dynamical equation of the matter can be expressed as

$$\ddot{\phi} + 3H\dot{\phi} = -V', \tag{3}$$

where $V = V(\phi)$ is a potential function and $V' = \frac{\partial V}{\partial \phi}$. The Hubble parameter satisfies

$$H^2 = \frac{1}{3M_{pl}^2}\left[\frac{1}{2}\dot{\phi}^2 + V\right]. \tag{4}$$

with M_{pl} is a constant called the reduced Planck mass. Inflation phase occurs during domination of potential energy V over kinetic energy $\dot{\phi}^2/2$ for sufficient time such that $\ddot{\phi}$ term in equation (3) can be neglected. This condition is called slow-roll approximation.

The quantities ϵ and $\eta_V = \delta + \epsilon$, where $\delta = -\ddot{\phi}/(H\dot{\phi})$, are called 'slow-roll' parameters.

C.1	Estimate parameter ϵ, parameter η_V, $dN/d\phi$ in terms of potential $V(\phi)$ and its first and second derivative (V' and V'').	1.7 pt.

D. Inflation with A Simple Potential

Predictions of any inflation model should be compared to observational constraints from CMB as follow n_s = 0.968 $\pm$ 0.006 and r < 0.12, where $r = 16\epsilon$ and $n_s = 1 + 2\eta_V - 6\epsilon$ are evaluated at $\phi = \phi_{start}$ for inflation model being generated by a dominant matter. Assume that potential of matter takes a simple form $V(\phi) = \Lambda^4 \left(\frac{\phi}{M_{pl}}\right)^n$ where n is any integer and Λ is a constant.

D.1	Calculate ϕ_{end} at the end of inflation.	0.5 pt.
D.2	Express r and n_s in terms of e-folding number N and integer n. Estimate the value of n that is closest to observational values r and n_s. Take N = 60 in your calculation.	0.9 pt.